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AN ELEMENTARY TREATISE

ON

CONIC SECTIONS.

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CONIC SECTIONS,

TREATED GEOMETRICALLY,

BY

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PREFACE.

In the present Treatise the Conic Sections are defined with reference to a focus and directrix, and I have endeavoured to place before the student the most important properties of those curves, deduced, as closely as possible, from the definition.

The construction which is given in the first Chapter for the determination of points in a conic section possesses several advantages; in particular, it leads at once to the constancy of the ratio of the square on the ordinate to the rectangle under its distances from the vertices; and, again, in the case of the hyperbola, the directions of the asymptotes follow immediately from the construction. In several cases the methods employed are the same as those of Wallace, in the Treatise on Conic Sections, published in the *Encyclopædia Metropolitana.*

The deduction of the properties of these curves from their definition as the sections of a cone, seems *à priori* to be the natural method of dealing with the subject, but experience appears to have shewn that the discussion of conics as defined by their plane properties is the most suitable method of commencing an elementary treatise, and accordingly I follow the fashion of the time in taking that order for the treatment of the subject. In Hamilton's book on *Conic Sections*, published in the middle of the last century, the properties of the cone are first considered, and the advantage of this method of commencing the subject, if the use of solid figures be not objected to, is especially shewn in the very general theorem of Art. (150). I have made much use of this treatise, and, in fact, it contains most of the theorems and problems which are now regarded as classical propositions in the theory of Conic Sections.

I have considered first, in Chapter I., a few simple properties of conics, and have then proceeded to the particular properties of each curve, commencing with the parabola, as in some respects, the simplest form of a conic section.

It is then shewn, in Chapter VI., that the sections of a cone by a plane produce the several curves in question, and lead at once to their definition as loci, and to several of their most important properties.

A chapter is devoted to the method of orthogonal projection, and another to the harmonic properties of curves, and to the relations of poles and polars, including the theory of reciprocal polars for the particular case in which the circle is employed as the auxiliary curve.

For the more general methods of projections, of reciprocation, and of anharmonic properties, the student will consult the treatises of Chasles, Poncelet, Salmon, Townsend, Ferrers, Whitworth, and others, who have recently developed, with so much fulness, the methods of modern Geometry.

I have to express my thanks to Mr R. B. Worthington, of St John's College, and of the Indian Civil Service, for valuable assistance in the constructions of Chapter XI., and also to Mr E. Hill, Fellow of St John's College, for his kindness in looking over the latter half of the proof sheets.

I venture to hope that the methods adopted in this treatise will give a clear view of the properties of Conic Sections, and that the numerous Examples appended to the various Chapters will be useful as an exercise to the student for the further extension of his conceptions of these curves.

W. H. BESANT.

CAMBRIDGE,
March, 1869.

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CONIC SECTIONS.

INTRODUCTION.

DEFINITION.

1. IF a straight line and a point be given in position in a plane, and if a point move in the plane in such a manner that its distance from the given point always bears the same ratio to its distance from the given line, the curve traced out by the moving point is called a Conic Section.

The fixed point is called the Focus, and the fixed line the Directrix of the conic section.

When the ratio is one of equality, the curve is called a Parabola.

When the ratio is one of less inequality, the curve is called an Ellipse.

When the ratio is one of greater inequality, the curve is called an Hyperbola.

These curves are called Conic Sections, because they can all be obtained from the intersections of a Cone by planes in different directions, a fact which will be proved hereafter.

It may be mentioned that a circle is a particular case of an ellipse, that two straight lines constitute a particular case of an hyperbola, and that a parabola may be looked upon as the limiting form of an ellipse or an hyperbola, under certain conditions of variation in the lines and magnitudes upon which those curves depend for their form.

The object of the following pages is to discuss the general forms and characters of these curves, and to determine their most important properties by help of the methods and relations developed in the first six books, and in the eleventh book of Euclid, and it will be found that, for this purpose, a knowledge of Euclid's Geometry is all that is necessary.

The series of demonstrations will shew the characters and properties which the curves possess in common, and also the special characteristics wherein they differ from each other; and the continuity with which the curves pass into each other will appear both from the definition of a conic section as a Locus, or curve traced out by a moving point, as well as from the fact that they are deducible from the intersections of a cone by a succession of planes.

CHAPTER I.

PROPOSITION I.

The Construction of a Conic Section.

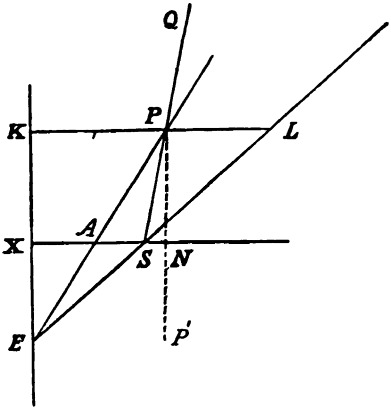
2. TAKE S as the focus, and from S draw SX at right angles to the directrix, and intersecting it in the point X.

DEFINITION. *This line SX, produced both ways, is called the Axis of the conic section.*

In SX take a point A such that the ratio of SA to AX is equal to the given ratio; then A is a point in the curve.

DEF. *The point A is called the Vertex of the curve.*

In the directrix EX take any point E, join EA, and ES, produce these lines, and through S draw the straight



line *SQ* making with *ES* produced the same angle which *ES* produced makes with the axis *SN*.

Let *P* be the point of intersection of *SQ* and *EA* produced, and through *P* draw *LPK* parallel to *NX*, and intersecting *ES* produced in *L*, and the directrix in *K*.

Then the angle *PLS* is equal to the angle *LSN* and therefore to *PSL*;

Hence $SP = PL$.

Also $PL : AS :: EP : EA$

$:: PK : AX$;

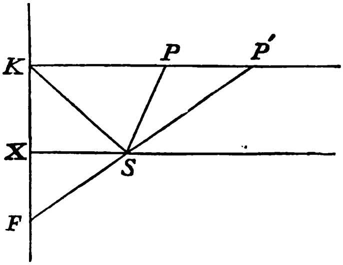
$\therefore PL : PK :: AS : AX$;

and $\therefore SP : PK :: AS : AX$.

The point *P* is therefore a point in the curve required, and by taking for *E* successive positions along the directrix we shall, by this construction, obtain a succession of points in the curve.

If *E* be taken on the upper side of the axis at the same distance from *X*, it is easy to see that a point *P* will be obtained below the axis, which will be similarly situated with regard to the focus and directrix. Hence it follows that the axis divides the curve into two similar and equal portions.

3. Another point of the curve, lying in the straight line *KP*, can be found in the following manner.



Through *S* draw the straight line *FS* making the angle *FSK* equal to *KSP* and let *FS* produced meet *KP* produced in *P′*.

Then, since KS bisects the angle PSF,

$$SP' : SP :: P'K : PK;$$

$$\therefore SP' : P'K :: SP : PK,$$

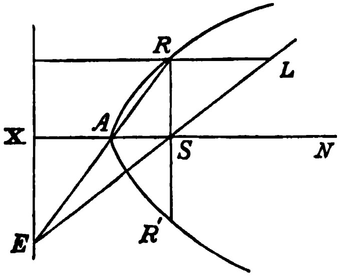
and P' is a point in the curve.

4. Def. *The Eccentricity. The constant ratio of the distance from the focus of any point in a conic section to its distance from the directrix is called the eccentricity of the conic section.*

The Latus Rectum. If E be so taken that EX is equal to SX, the angle PSN, which is double the angle LSN, and therefore double the angle ESX, is a right angle.

For, since $EX=SX$, the angle $ESX=SEX$, and, the angle SXE being a right angle, the sum of the two angles SEX, ESX, which is equal to twice ESX, is also equal to a right angle.

Calling R the position of P in this case produce RS to R', so that $R'S=RS$; then R' is also a point in the curve.



Def. *The straight line RSR' drawn through the focus at right angles to the axis, and intersecting the curve in R and R', is called the Latus Rectum.*

It is hence evident that the form of a conic section is determined by its eccentricity, and that its magnitude is determined by the magnitude of the Latus Rectum.

5. Def. *The straight line PN* (Fig. Art. 2), *drawn from any point P of the curve at right angles to the axis, and intersecting the axis in* N, *is called the Ordinate of the point P.*

If the line PN be produced to P' so that $NP' = NP$, the line PNP' is a *double ordinate* of the curve.

The latus rectum is therefore the double ordinate passing through the focus.

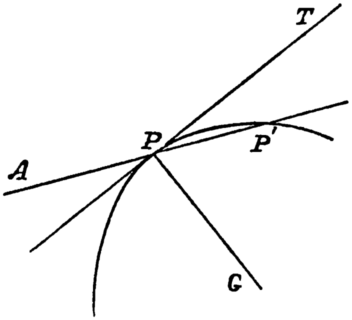
Def. *The distance AN of the foot of the ordinate from the vertex is called the Abscissa of the point P.*

Def. *The distance SP is called the focal distance of the point P.*

It is also described as the radius vector drawn from the focus.

6. *Definition of the Tangent to a curve.*

If a straight line, drawn through a point P of a curve, meet the curve again in P', and if the straight line be turned round the point P until the point P' approaches indefinitely near to P, the ultimate position of the straight line is the tangent to the curve at P.



Thus, if the straight line APP' turn round P until the points P and P' coincide, the line in its ultimate position PT is the tangent at P.

Def. *The normal at any point of a curve is the straight line drawn through the point at right angles to the tangent at that point.*

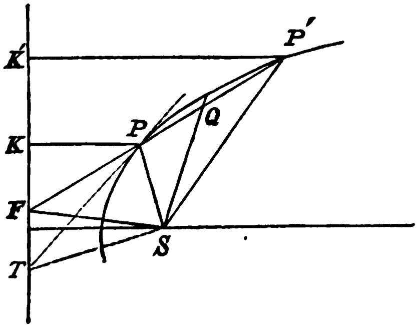
Thus, in the figure, PG is the normal at P.

7. We have now given a general method of constructing a conic section, and we have explained the nomenclature

which is usually employed. We proceed to demonstrate a few of the properties which are common to all the conic sections.

For the future the word conic will be employed as an abbreviation for conic section.

PROP. II. *If the straight line joining two points* P, P' *of a conic meet the directrix in* F, *the straight line* FS *will bisect the angle between* PS *and* $P'S$ *produced.*



Draw the perpendiculars PK, $P'K'$, on the directrix.

$$\text{Then} \qquad SP : SP' :: PK : P'K'$$
$$:: PF : P'F.$$

Therefore FS bisects the outer angle, at S, of the triangle PSP'. (Euclid, VI. A.)

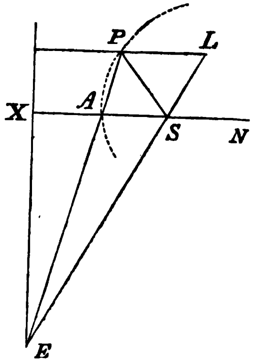
If SQ bisect the angle PSP', it follows that FSQ is a right angle.

Let the point P' move along the curve towards P; then, as P' approaches to coincidence with P, the straight line FP approximates to, and ultimately becomes, the tangent TP at P. But, when P' coincides with P, the line SQ coincides with SP, and the angle FSP which is ultimately TSP, becomes a right angle.

Hence the straight line, drawn from the focus to the point in which the tangent meets the directrix, is at right angles to the straight line drawn from the focus to the point of contact.

Or, in other words, the portion of the tangent intercepted between the point of contact and the directrix, subtends a right angle at the focus.

8. If a chord EAP be drawn through the vertex, and the point P be near the vertex, the angle PSA is small, and LSN which is half the angle PSN is nearly a right angle. The angle ASE is therefore nearly a right angle, and SEX is a small angle, and AES is, *a fortiori*, a small angle, and vanishes when ASE is a right angle.



As P approaches to coincidence with A, the angle LSN becomes ultimately a right angle, and therefore ASE is ultimately a right angle.

Hence the angle EAX which is the sum of the angles AES, ASE, is a right angle when P coincides with A.

But, when P approaches to coincidence with A, then EAP approximates to the position of, and ultimately becomes, the tangent at A.

The tangent at the vertex is therefore at right angles to the axis.

9. Prop. III. *No straight line can meet a conic in more than two points.*

In the figure of Prop. II, let P be a point in the curve, draw any straight line FP, join SF, and draw SQ at right angles to SF. Draw SP' making the angle QSP' equal to QSP; then P' is a point in the curve. For, since SF bisects the outer angle at S,

$$SP' : SP :: P'F : PF$$
$$:: P'K' : PK;$$
$$\therefore SP' : P'K' :: SP : PK,$$

and P' is a point in the curve.

Also, there is no other point of the curve in the straight ne FP.

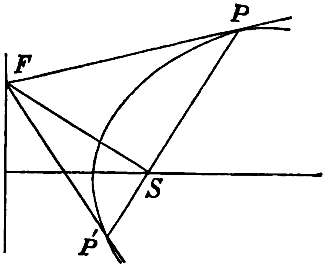
For suppose if possible P'' to be another point and draw $P''K''$ perpendicular to the directrix,

then $$SP'' : SP :: P''K'' : PK$$
$$:: P''F : PF;$$

therefore FS bisects the angle between PS and $P''S$ produced.

But FS bisects the angle between PS and $P'S$ produced, which is impossible unless P'' coincides with P'.

10. Prop. IV. *The tangents at the ends of a focal chord intersect in the directrix.*

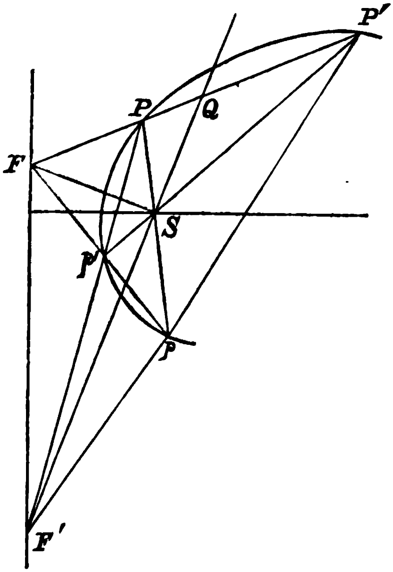


For the line SF perpendicular to SP, meets the directrix in the same point as the tangent at P; and, since FS is also at right angles to SP', the tangent at P' meets the directrix in the same point F. Conversely, if from any point F in the directrix tangents be drawn, the chord of contact, that is, the straight line joining the points of contact, will pass through the focus and will be at right angles to SF.

Cor. Hence it follows that the tangents at the ends of the latus-rectum pass through the foot of the directrix.

11. Prop. V. *The straight lines joining the extremities of two focal chords intersect in the directrix.*

If PSp, $P'Sp'$ be the two chords, the point in which PP' meets the directrix is obtained by bisecting the angle PSP' and drawing SF at right angles to the bisecting line SQ.



But this line also bisects the angle pSp'; therefore pp' also passes through F.

The line SF bisects the angle PSp', and similarly, if QS produced, bisecting the angle pSp', meet the directrix in F', the two lines Pp', $P'p$ will meet in F'.

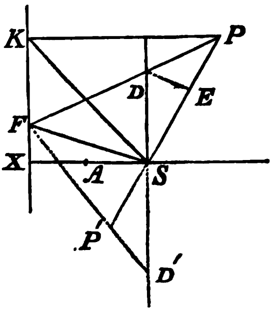
It is obvious that the angle FSF' is a right angle.

Corollary. If the straight line bisecting PSP' meet the curve in q and q' and Fq, Fq' be joined, these lines will be the tangents at q and q'. (Prop. IV.)

Hence, if from a point F in the directrix tangents be drawn, and also any straight line FPP' cutting the curve in P and P', the chord of contact will bisect the angle PSP'.

12. Prop. VI. *If the tangent at any point P of a conic intersect the directrix in F, and the latus rectum produced in D,*

$$SD : SF :: SA : AX.$$



Join SK; then, observing that FSP and FKP are right angles, a circle can be described about $FSPK$, and therefore the angles SFD, SKP are equal.

Also the angle FSD
$=$ complement of DSP
$= SPK$;

$\therefore$ the triangles FSD, SPK are similar, and

$$SD : SF :: SP : PK$$
$$:: SA : AX.$$

Cor. (1). If DE be drawn at right angles to SP, SE is equal to the semi-latus rectum.

For SDE, SFX are similar triangles, and

$$SE : SX :: SD : SF$$
$$:: SA : AX$$
$$:: SR : SX.$$

SE is therefore equal to SR, the semi-latus rectum.

Cor. (2). If the tangents at the ends of a focal chord meet the latus rectum in D and D',

$$SD = SD'.$$

For the tangents meet the directrix in the same point F, and, by the same reasoning as in Prop. VI,

$$SD' : SF :: SA : AX;$$
$$\therefore \quad SD = SD'.$$

13. Prop. VII. *The tangents drawn from any point to a conic subtend equal angles at the focus.*

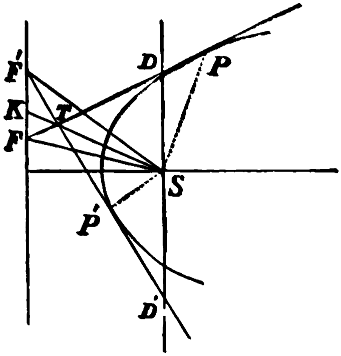
Let the tangents FTP, $F'TP'$ at P and P' meet the directrix in F and F' and the latus rectum in D and D'.

Join ST and produce it to meet the directrix in K;

then $$KF : SD :: KT : ST$$
$$:: KF' : SD'.$$

Hence $$KF : KF' :: SD : SD'$$
$$:: SF : SF' \text{ by Prop. VI};$$

$\therefore$ the angles TSF, TSF' are equal.



But the angles FSP', $F'SP$ are equal, for each is the complement of FSF';

$\therefore$ the angles TSP, TSP' are equal.

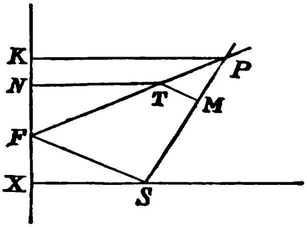
Cor. Hence it follows that if perpendiculars, TM, TM' be let fall upon SP and SP', they are equal in length.

For the two triangles TSM, TSM' have the angles TMS, TSM respectively equal to the angles $TM'S$ TSM', and the side TS common; and therefore the other sides are equal,

and $$TM = TM'.$$

14. Prop. VIII. *If from any point T in the tangent at a point P of a conic, TM be drawn perpendicular to the focal distance SP, and TN perpendicular to the directrix,*

$$SM : TN :: SA : AX.$$



For, if PK be perpendicular to the directrix and SF be joined,

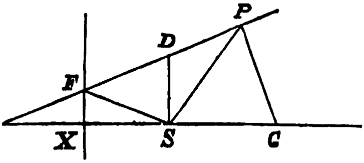
$$\begin{aligned} SM : SP &:: TF : FP \\ &:: TN : PK; \\ \therefore\ SM : TN &:: SP : PK \\ &:: SA : AX. \end{aligned}$$

This proposition, which is due to Professor Adams, may also be employed to prove Prop. VII, and, moreover, it suggests an obvious method of drawing tangents from any point to a conic.

For, if T be the point, and a circle be described about S as centre, the radius of which bears to TN the ratio of $SA : AX$, and if tangents TM, TM' be drawn to the circle the straight lines SM, SM', produced if necessary, will intersect the conic in the points of contact of the tangents from T.

15. PROP. IX. *If PG the normal at P meet the axis of the conic in G,*

$$SG : SP :: SA : AX.$$



Let the tangent at P meet the directrix in F, and the latus rectum produced in D.

Then the angle SPG = the complement of SPF

$$= PFS,$$

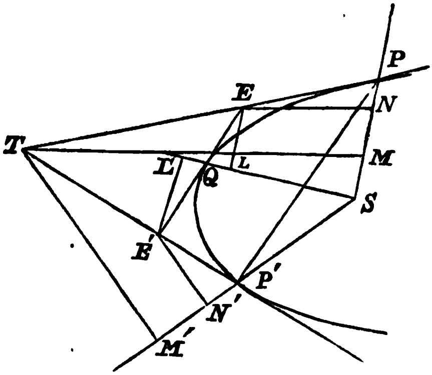
and the angle PSG = the complement of FSX

$$= FSD;$$

$\therefore$ the triangles SDF, SPG are similar, and

$$\begin{aligned} SG : SP &:: SD : SF \\ &:: SA : AX, \text{ by Prop. VI.} \end{aligned}$$

16. **Prop. X.** ***If a tangent be drawn parallel to a chord of a conic, the portion of this tangent which is intercepted by the tangents at the ends of the chord is bisected at the point of contact.***



Let PP' be the chord, TP, TP' the tangents, and EQE' the tangent parallel to PP'.

From the focus S draw SP, SP' and SQ, and draw TM, TM' perpendicular respectively to SP, SP'.

Also draw from E perpendiculars EN, EL, upon SP, SQ, and from E' perpendiculars $E'N'$, $E'L'$ upon SP' and SQ.

Then, since EE' is parallel to PP'

$$TE : EP :: TE' : E'P',$$

but $\quad TE : EP :: TM : EN,$

and $\quad TE' : E'P' :: TM' : E'N';$

$$\therefore\ TM : EN :: TM' : E'N';$$

but $TM = TM'$, Cor. Prop. VII;

$$\therefore\ EN = E'N'.$$

Again, by the same corollary,

$$EN = EL \text{ and } E'N' = E'L';$$

$$\therefore\ EL = E'L',$$

and, the triangles ELQ, $E'L'Q$ being similar,

$$EQ = E'Q.$$

Cor. If TQ be produced to meet PP' in V,

$$PV : EQ :: TV : TQ,$$

and
$$P'V : E'Q :: TV : TQ;$$

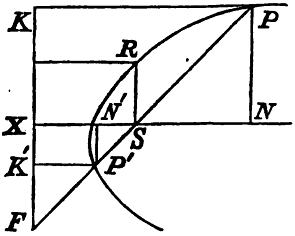
$$\therefore PV = P'V,$$

that is, PP' is bisected in V.

Hence, if tangents be drawn at the ends of any chord of a conic, the point of intersection of these tangents, the middle point of the chord, and the point of contact of the tangent parallel to the chord all lie in one straight line.

17. Prop. XI. *The semi-latus rectum is an harmonic mean between the two segments of any focal chord of a conic.*

Def. Three magnitudes are said to be in harmonic progression when the first is to the third as the difference between the first and second is to the difference between the second and third: and the second magnitude is said to be an harmonic mean between the first and third.



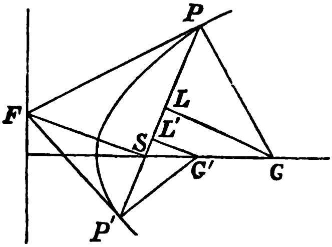
Let PSP' be a focal chord, and draw the ordinates PN, $P'N'$.

Then the triangles SPN, $SP'N'$ are similar;

$$\begin{aligned} \therefore SP : SP' &:: SN : SN' \\ &:: NX - SX : SX - N'X \\ &:: SP - SR : SR - SP', \end{aligned}$$

since SP, SR, SP' are proportional to NX, SX, and SP'.

18. Prop. XII. *If from G, the point in which the normal at P meets the axis, GL be drawn perpendicular to SP, the length PL is equal to the semi-latus rectum.*



Produce PS to meet the curve in P'; let F be the point of intersection of the tangents at P and P', and join SF.

Draw the normals PG, $P'G'$, and draw GL, $G'L'$ at right angles to PSP'.

Then LPG, SPF are similar triangles, and

$$PL : SF :: LG : SP.$$

Similarly $$P'L' : SF :: L'G' : SP'.$$

But $LG : L'G' :: SG : SG'$

$$:: SP : SP' \text{ by Prop. IX.}$$

$$\therefore LG : SP :: L'G' : SP'.$$

Hence $$PL : SF :: P'L' : SF,$$

and $$PL = P'L'.$$

Again, $SP : SP' :: SG : SG'$

$$:: SL : SL'$$

$$:: SP - PL : PL - SP',$$

and PL is therefore an harmonic mean between SP and SP'.

Hence, Prop. XI, PL is equal to the semi-latus rectum.

19. Prop. XIII. *A focal chord is divided harmonically at the focus and the point where it meets the directrix.*

Let PSP' produced meet the directrix in F, and draw PK, $P'K'$ perpendicular to the directrix, fig. art. 17.

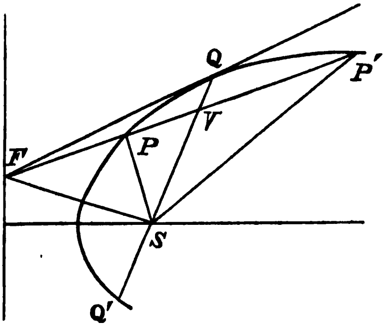
Then $PF : P'F :: PK : P'K'$

$:: SP : SP'$

$:: PF - SF : SF - P'F;$

that is, PF, SF, and $P'F$ are in harmonic progression, and the line PP' is divided harmonically at S and F.

20. Prop. XIII. *If from any point F in the directrix tangents be drawn, and also any straight line FPP' cutting the curve in P and P', the chord PP' is divided harmonically at F and its point of intersection with the chord of contact.*



For, if QSQ' be the chord of contact, it bisects the angle PSP', (corollary, Prop. V.), and ∴, if V be the point of intersection of SQ and PP',

$$FP' : FP :: SP' : SP$$
$$:: P'V : PV$$
$$:: FP' - FV : FV - FP.$$

Hence FV is an harmonic mean between FP and FP'.

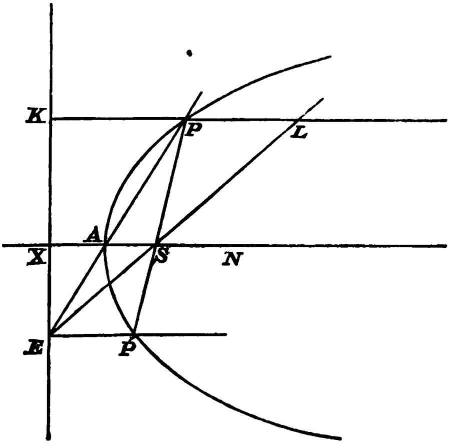
The last two theorems are particular cases of more general theorems, which will appear hereafter.

CHAPTER II.

THE PARABOLA.

DEF. *A parabola is the curve traced out by a point which moves in such a manner that its distance from a given point is always equal to its distance from a given straight line.*

Tracing the Curve.



21. Let S be the focus, EX the directrix, and SX the perpendicular on EX. Then, bisecting SX in A, the point A is the vertex; and, if from any point E in the directrix, EAP, ESL be drawn, and from S the straight line SP meeting EA produced in P, and making the angle PSL equal to LSN, we obtain {as in Art. (2)}, a point P in the curve.

For $\qquad PL : PK :: SA : AX,$

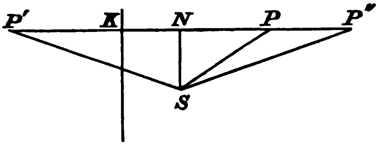
and $\qquad \therefore PL = PK.$

But $SP = PL$, and $\therefore SP = PK$.

Again, drawing EP' parallel to the axis and meeting in P' the line PS produced, we obtain the other extremity of the focal chord PSP'.

For the angle
$$ESP' = PSL = PLS$$
$$= SEP',$$
and
$$\therefore SP' = P'E,$$
and P' is a point in the parabola.

The curve lies wholly on the same side of the directrix;



for, if P' be a point on the other side, and SN be perpendicular to $P'K$, SP' is greater than $P'N$, and therefore is greater than $P'K$.

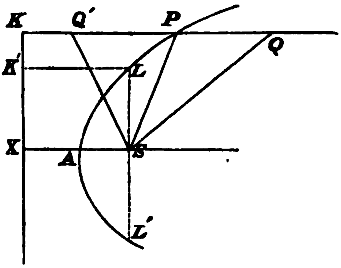
Again, a straight line parallel to the axis meets the curve in one point only.

For, if possible, let P'' be another point of the curve in KP produced.

Then
$$SP = PK \text{ and } SP'' = P''K;$$
$$\therefore PP'' = SP'' - SP,$$
or
$$PP'' + SP = SP'',$$
which is impossible.

Lastly, the curve has infinite branches; for, since $PS = PK = PL$, it follows that P is the centre of the circle passing through K, S, and L, and therefore the angle KSL in a semicircle is a right angle; hence it follows that, as E approaches X, the point K moves away from X, and therefore the point P moves away from the axis, its distance becoming larger as the distance EX diminishes. Since ESK is a right angle the rectangle $EX . KX$ is equal to SX^2, and therefore when EX is indefinitely small, KX is indefinitely large. The curve therefore has two branches proceeding to infinity.

22. Prop. I. *The distance from the focus of a point inside a parabola is less, and of a point outside is greater, than its distance from the directrix.*



If Q be the point inside, let fall the perpendicular QPK on the directrix, meeting the curve in P.

Then $SP+PQ>SQ$, but $SP+PQ$

$$=PK+PQ=QK,$$
$$\therefore\ SQ<QK.$$

If Q' be outside, and between P and K,

$$SQ'+PQ'>SP,$$
$$\therefore\ SQ'>Q'K.$$

If Q' lie in PK produced,

$$SQ'+SP>PQ',$$

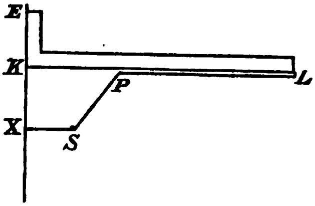
and $$\therefore\ SQ'>KQ'.$$

23. Prop. II. *The Latus Rectum* $=4 . AS$.

For if, Fig. Art. 22, LSL' be the Latus Rectum, drawing LK' at right angles to the directrix, we have

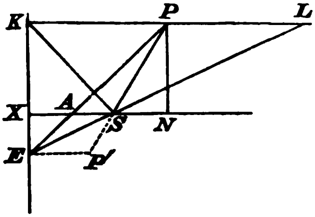
$$LS=LK'=SX=2AS,$$
$$\therefore\ LSL'=4 . AS.$$

24. *Mechanical construction of the Parabola.*



Take a rigid bar EKL, of which the portions EK, KL are at right angles to each other, and fasten a string to the end L, the length of which is LK. Then if the other end of the string be fastened to S, and the bar be made to slide along the directrix, a pencil at P, keeping the string stretched against the bar, will trace out a portion of a parabola.

25. **Prop. III.** *If PN be the ordinate of a point P,* $PN^2 = 4AS.AN$.



Draw the lines PAE, ESL, KPL, and join SK.

Then $SP = PL = PK$. Hence KSL, and $\therefore KSE$, is a right angle, and $EX.KX = SX^2 = 4AS^2$; also

$$AN : AX :: PN : EX$$
$$:: PN^2 : EX.KX,$$
$$:: PN^2 : 4AS^2;$$
$$\therefore\ PN^2 = 4AS.AN.$$

26. Conversely, if it be known that at every point of a curve the relation $PN^2 = 4AS.AN$ holds true, the curve is a parabola.

In NA produced take AX equal to AS, draw EXK at right angles to XN, and KPL parallel to XN; also draw PAE, and ESL.

Then $$AN : AX :: PN : EX,$$
or $$AN : AS :: PN^2 : EX.KX,$$
but $$PN^2 = 4AS.AN;$$
$$\therefore\ EX.KX = 4AS^2 = SX^2.$$

Hence KSE, and $\therefore KSL$, is a right angle,

and, since $$SA = AX,\ PL = PK;$$
$$\therefore\ SP = PK,$$
which is the definition of a parabola.

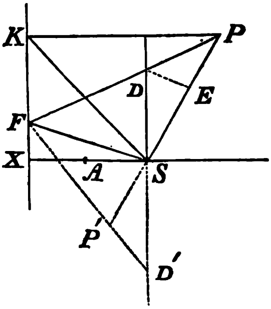
27. **Prop. IV.** *If from the ends of a focal chord perpendiculars be let fall upon the directrix, the intercepted portion of the directrix subtends a right angle at the focus.*

For, if the straight line through E parallel to the axis meet PS in P', P' is the other extremity of the focal chord PS, and, as in Art. 25, KSE is a right angle.

Cor. Since ES bisects the angle ASP', Art. 21, it follows that KS bisects the angle ASP.

28. Prop. V. *The tangent at any point P bisects the angle between the focal distance SP and the perpendicular PK on the directrix.*

Let F be the point in which the tangent meets the directrix, and join SF.



We have shewn, Arts. 7 and 12, that FSP is a right angle, and that, if the latus-rectum produced meet the tangent in D,

$$SD : SF :: SA : AX;$$

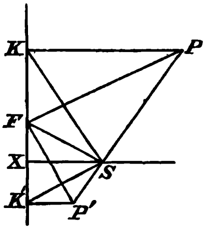
but $SA = AX$, and

$$\therefore SD = SF.$$

Hence the angle $SFP = SDF = PFK$, and therefore the complementary angles SPF, FPK are equal.

Cor. It has been shewn, in Art. (10), that the tangents at the ends of a focal chord intersect in the directrix, and therefore, if PS produced meet the curve in P', FP' is the tangent at P', and bisects the angle between SP' and the perpendicular from P' on the directrix.

29. Prop. VI. *The tangents at the ends of a focal chord intersect at right angles in the directrix.*



Let PSP' be the chord, and PF, $P'F$ the tangents meeting the directrix in F.

Let fall the perpendiculars PK, PK', and join SK, SK'.

The angle $P'SK' = \frac{1}{2}P'SX$
$= \frac{1}{2}SPK = SPF$,

$\therefore$ SK is parallel to PF,

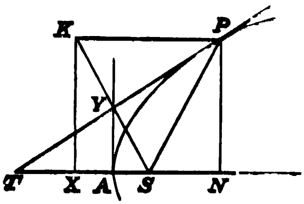
and, similarly, SK is parallel to $P'F$.

But KSK is a right angle;

$\therefore$ PFP' is a right angle.

30. **Prop. VII.** *If the tangent at any point P of a parabola meet the axis in T, and PN be the ordinate of P, then*

$$AT = AN.$$



Draw PK perpendicular to the directrix.

The angle SPT

$$= TPK$$
$$= PTS,$$
$$\therefore ST = SP$$
$$= PK$$
$$= NX.$$

But $$ST = SA + AT,$$
and $$NX = AN + AX;$$
$$\therefore \text{ since } SA = AX,$$
$$AT = AN.$$

Def. *The line NT is called the sub-tangent.*

The sub-tangent is therefore twice the abscissa of the point of contact.

31. **Prop. VIII.** *The foot of the perpendicular from the focus on the tangent at any point P of a parabola lies on the tangent at the vertex, and the perpendicular is a mean proportional between SP and SA.*

Taking the figure of the previous article, join SK meeting PT in Y.

Then $SP = PK$, and PY is common to the two angles SPY, KPY;

also the angle $SPY = YPK$;

$\therefore$ the angle $SYP = PYK$,

and SY is perpendicular to PT.

Also $SY = KY$, and $SA = AX$, $\therefore SY : YK :: SA : AX$, and AY is parallel to KX.

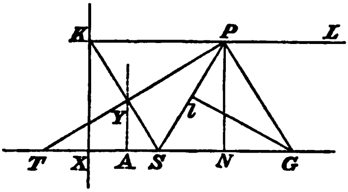
Hence, AY is at right angles to AS, and is therefore the tangent at the vertex.

Again, the angle $SPY = STY = SYA$, and the triangles SPY, SYA are therefore similar;

$$\therefore SP : SY :: SY : SA,$$
$$\text{or } SY^2 = SP \,.\, SA.$$

32. **Prop. IX.** *In the parabola the subnormal is constant and equal to the semi-latus Rectum.*

Def. *The distance between the foot of the ordinate of P, and the point in which the normal at P meets the axis is called the subnormal.*



In the figure PG is the normal and PT the tangent.

It has been shewn that the angle SPK is bisected by PT, and hence it follows that SPL is bisected by PG,

and that the angle $SPG = GPL = PGS$;

hence
$$\begin{aligned} SG &= SP = ST \\ &= SA + AT = SA + AN \\ &= 2AS + SN; \end{aligned}$$

$\therefore$ the subnormal $NG = 2AS$.

33. Cor. If Gl be drawn perpendicular to SP,

$$\begin{aligned} \text{the angle } GPl &= \text{the complement of } SPT, \\ &= \text{the complement of } STP, \\ &= PGN, \end{aligned}$$

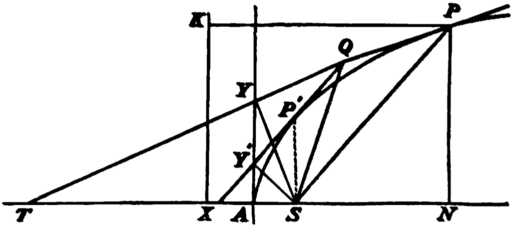
and the two right-angled triangles GPN, GPl have their angles equal and the side GP common; hence the triangles are equal, and

$$\begin{aligned} Pl &= NG = 2AS \\ &= \text{the semi-latus Rectum.} \end{aligned}$$

It has been already shewn, Art. (18), that this property is a general property of all conics.

34. Prop. X. *To draw tangents to a parabola from an external point.*

For this purpose we may employ the general construction given in Art. (14), or, for the special case of the parabola, the following construction.



Let Q be the external point, join SQ, and upon SQ as diameter describe a circle intersecting the tangent at the vertex in Y and Y'. Join YQ, $Y'Q$; these are tangents to the parabola.

Draw SP, so as to make the angle YSP equal to YSA, and to meet YQ in P, and let fall the perpendicular PN upon the axis.

Then, SYQ is a right angle, since it is the angle in a semicircle, and, T being the point in which QY produced meets the axis, the two triangles SYP, SYT are equal in all respects;

$$\therefore\ SP = ST, \text{ and } YT = YP.$$

But AY is parallel to PN;

$$\therefore\ AT = AN.$$

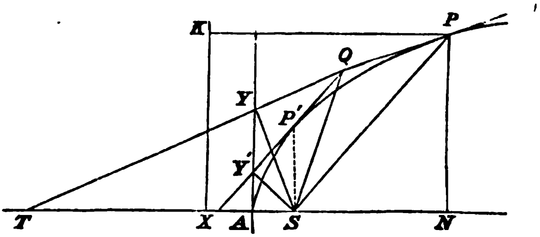
Hence
$$\begin{aligned} SP &= ST = SA + AT \\ &= AX + AN \\ &= NX, \end{aligned}$$

and P is a point in the parabola.

Moreover, if PK be perpendicular to the directrix, the angle
$$SPY = STP = YPK,$$
and PY is the tangent at P. Art. (28).

Similarly, by making the angle $Y'SP'$ equal to ASY', we obtain the point of contact of the other tangent QY'.

35. **Prop. XI.** *If from a point Q tangents QP, QP' be drawn to a parabola, the two triangles SPQ, SQP' are similar, and SQ is a mean proportional between SP and SP'.*



Produce PQ to meet the axis in T, and draw SY, SY' perpendicularly on the tangents. Then Y and Y' are points in the tangent at A.

The angle
$$\begin{aligned} SPQ &= STY \\ &= SYA \\ &= SQP', \end{aligned}$$

since $SY'YQ$ are points on a circle, and SYA, SQP' are in the same segment.

Also, by the theorem of Art. (13), the angle

$$PSQ = QSP';$$

therefore the triangles PSQ, QSP' are similar, and

$$SP : SQ :: SQ : SP'.$$

36. From the theorem of Art. 35 the following, which is often useful, immediately follows.

If from any points in a given tangent of a parabola, tangents be drawn to the curve, the angles which these tangents make with the focal distances of the points from which they are drawn are all equal.

For each of them, by the theorem, is equal to the angle between the given tangent and the focal distance of the point of contact.

37. Since the two triangles PSQ, QSP' are similar, we have

$$PQ : P'Q :: SP : SQ$$

and
$$PQ : P'Q :: SQ : SP',$$

$$\therefore PQ^2 : P'Q^2 :: SP : SP';$$

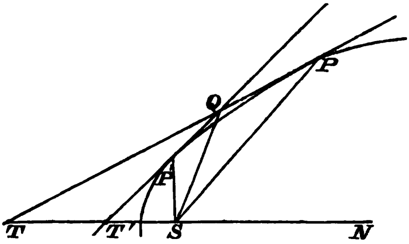
that is, the squares of the tangents from any point are proportional to the focal distances of the points of contact.

This will be found to be a particular case of a subsequent Theorem.

38. PROP. XII. *The external angle between two tangents is half the angle subtended at the focus by the chord of contact.*

Let the tangents at P and P' intersect each other in Q, and the axis ASN in T and T'.

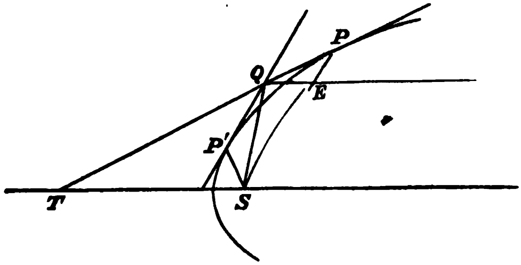
Join SP, SP'; then the angles SPT, STP are equal, and $\therefore STP$ is half the angle PSN; similarly $ST'P'$ is half $P'SN$.



But TQT' is equal to the difference between STP and $ST'P'$, and is therefore equal to half the difference between PSN and $P'SN$, that is to half the angle PSP'.

Hence, joining SQ, TQT' is equal to each of the angles PSQ, $P'SQ$.

39. **Prop. XIII.** *The tangents drawn to a parabola from any point make the same angles, respectively, with the axis and the focal distance of the point.*



Let QP, QP' be the tangents; join SP, and draw QE parallel to the axis, and meeting SP in E.

Then, if PQ meet the axis in T, the angle

$$EQP = STP = SPQ$$

$$= SQP'. \quad \text{Art. (36).}$$

i. e. QP and QP' respectively make the same angles with the axis and with QS.

40. Conceive a parabola to be drawn passing through Q, having S for its focus, SN for its axis, and its vertex on the same side of S as the vertex A of the given parabola. Then the normal at Q to this new parabola bisects the angle SQE; therefore the angles which QP and QP' make with the normal at Q are equal.

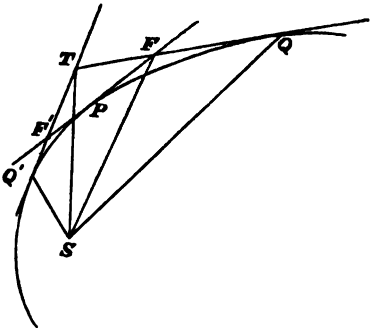
Hence the theorem,

If from any point in a parabola, tangents be drawn to a confocal and co-axial parabola, the normal at the point will bisect the angle between the tangents.

In this enunciation the words co-axial and confocal are intended to imply, not merely the coincidence of the axes, but also that the vertices of the two parabolas are on the same side of their common focus.

The reason for this will appear when we shall have discussed the analogous property of the ellipse.

41. Prop. XIV. *The circle passing through the points of intersection of three tangents passes also through the focus.*



Let Q, P, Q' be the three points of contact, and F, T, F' the intersections of the tangents.

In Art. (35) it has been shewn that, if FP, FQ be tangents, the angle

$$SQF = SFP.$$

Similarly TQ, TQ' being tangents, the angle

$$SQT = STQ',$$

hence the angle SFF' or $SFP = SQT$,

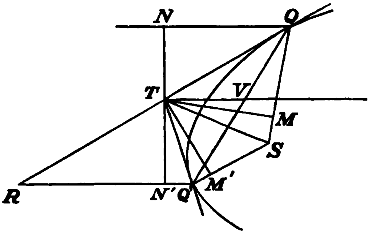
$$= STF',$$

and a circle can be drawn through S, F, T, and F'.

42. Def. *A straight line drawn parallel to the axis through any point of a parabola is called a diameter.*

PROP. XV. *If from any point T tangents TQ, TQ′ be drawn to a parabola, the point T is equidistant from the diameters passing through Q and Q′, and the diameter drawn through the point T bisects the chord of contact.*

Join SQ, SQ', and draw TM, TM' perpendicular respectively to SQ and SQ'.



Also draw NTN' perpendicular to the diameters through Q and Q', and meeting those diameters in N and N'.

Then, since TS bisects the angle QSQ',

$$TM = TM';$$

and, since TQ bisects the angle SQN,

$$TN = TM.$$

Similarly $$TN' = TM',$$

$$\therefore\ TN = TN'.$$

Again, join QQ', and draw the diameter TV meeting QQ' in V; also let QT produced meet $Q'N'$ in R;

then $$QV : VQ' :: QT : TR$$
$$:: TN : TN',$$

since the triangles QTN, RTN' are similar;

$$\therefore\ QV = VQ'.$$

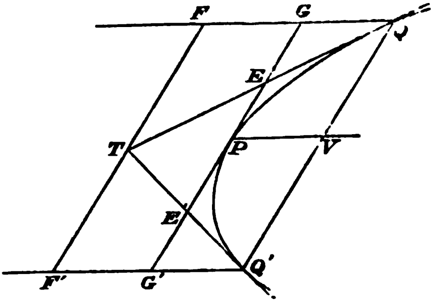
Hence *the diameter through the middle point of a chord passes, when produced, through the point of intersection of the tangents at the ends of the chord.*

It should be noticed that any straight line drawn through T and terminated by QN and $Q'N'$ is bisected at T.

43. Prop XV. ***Any diameter bisects all chords parallel to the tangent at its extremity, and passes through the point of intersection of the tangents at the ends of any of these chords.***

Let QQ' be a chord parallel to the tangent at P, and through the point of intersection T of the tangents at Q and Q' draw FTF' parallel to QQ' and terminated at F and F' by the diameters through Q and Q'.

Let the tangent at P meet TQ, TQ' in E and E', and QF, $Q'F'$ in G and G'.



Then $$EG : TF :: EQ : TQ$$
$$:: E'Q' : TQ'$$
$$:: E'G' : TF'.$$

But $TF = TF'$, since, Art. (42), T is equidistant from QG and $Q'G'$,

$$\therefore EG = E'G'.$$

also, $EP = EG$, since E is equidistant from QG and PV,

$$\therefore EP = E'P \text{ and } GP = PG'.$$

Hence PV being the diameter at P,

$$QV = VQ'.$$

Again, since T, P, V are each equidistant from the parallel straight lines QF, $Q'F'$, it follows that TPV is a straight line, or that the diameter VP passes through T.

We have shewn that GE, EP, PE', $E'G'$ are all equal, and we hence infer that

$$EE' = \tfrac{1}{2}\, GG' = \tfrac{1}{2}\, QQ',$$

and consequently that $TP = \frac{1}{2}\, TV$ or that $TP = PV$.

Hence *it appears, that the diameter through the point of intersection of a pair of tangents passes through the point of contact of the tangent parallel to the chord of contact, and also through the middle point of the chord of contact; and that the portion of the diameter between the point of intersection of the tangents and the middle point of the chord of contact is bisected at the point of contact of the parallel tangent.*

We may observe that in proving that EE' is bisected at P, we have demonstrated a theorem already shewn, Art. (16), to be true for all conics.

44. DEF. *The line QV, parallel to the tangent at P, and terminated by the diameter PV, is called an ordinate of that diameter, and QQ' is the double ordinate. The point P, the end of the diameter, is called its vertex.*

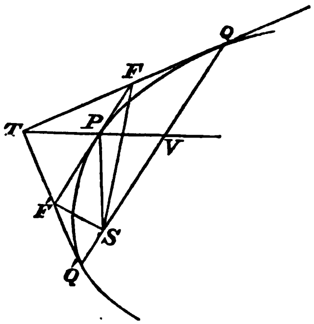
We observe that tangents at the ends of any chord intersect in the diameter which bisects the chord, and that the distance of this point from the vertex is equal to the distance of the vertex from the middle point of the chord.

DEF. *The chord through the focus parallel to the tangent at any point is called the parameter of the diameter passing through the point.*

45. PROP. XVI. *The parameter of any diameter is four times the focal distance of the vertex of that diameter.*

Let P be the vertex, and QSQ' the parameter, T the point of intersection of the tangents at Q and Q', and FPF' the tangent at P.

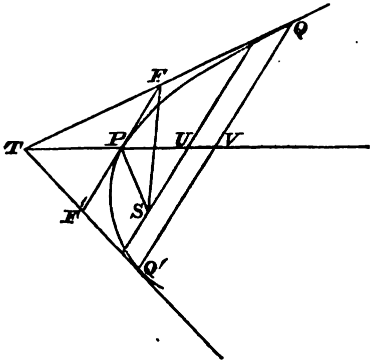
Then, since FS and $F'S$ bisect respectively the angles



PSQ, PSQ', FSF' is a right angle, and, P being the middle point of FF', $SP = PF = PF'$.

Hence QQ', which is double FF', is four times SP.

46. Prop. XVII. *If* QVQ' *be a double ordinate of a diameter* PV, QV *is a mean proportional between* PV *and the parameter of* P.



Let FPF' be the tangent at P, and draw the parameter through S meeting PV in U.

The angle $SUT = FPU = SPF' = SFT$, Art. 36, and therefore U is a point in the circle which can be drawn through $SFTF'$.

Hence, QV being twice PF,

$$QV^2 = 4PF^2 = 4PU.PT;$$

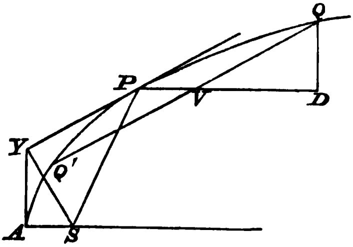
but $$PU = SP,$$

for the angle $SUP = FPU = SPF' = PSU$;

and $$PT = PV,$$

$$\therefore QV^2 = 4SP.PV.$$

47. **Prop. XVIII.** *If QVQ' be a double ordinate of a diameter PV, and QD the perpendicular from Q upon PV QD is a mean proportional between PV and the latus rectum.*



Let the tangent at P meet the tangent at the vertex in Y, and join SY.

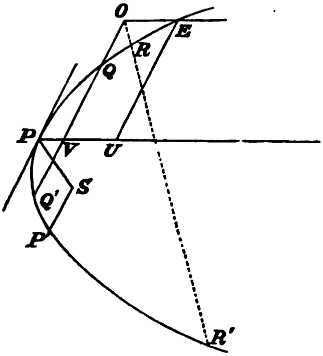
The angle $QVD = SPY = SYA$, and therefore the triangles QVD, SAY are similar;

and $$QD^2 : QV^2 :: AS^2 : SY^2$$
$$:: AS^2 : AS.SP$$
$$:: AS : SP$$
$$:: 4AS.PV : 4SP.PV,$$

but $$QV^2 = 4SP.PV;$$

$$\therefore QD^2 = 4AS.PV.$$

48. **Prop. XIX.** *If from any point, within or without a parabola, two straight lines be drawn in given directions and intersecting the curve, the ratio of the rectangles of the segments is independent of the position of the point.*



From any point O draw a straight line intersecting the parabola in Q and Q', and draw the diameter OE, meeting the curve in E.

Let PV be the diameter bisecting QQ', and draw the ordinate EU.

Then
$$\begin{aligned} OQ \,.\, OQ' &= OV^2 - QV^2 \\ &= EU^2 - QV^2 \\ &= 4SP \,.\, PU - 4SP \,.\, PV \\ &= 4SP \,.\, OE. \end{aligned}$$

Similarly, if ORR' be any other intersecting line and P' the vertex of the diameter bisecting RR',

$$OR \,.\, OR' = 4SP' \,.\, OE.$$

Therefore

$$OQ \,.\, OQ' : OR \,.\, OR' :: SP : SP',$$

that is, the ratio of the rectangles depends only on the positions of P and P', and, if the lines OQQ', ORR' are drawn parallel to given straight lines, these points P, P' are fixed.

It will be easily seen that the proof is the same if the point O be within the parabola.

If the lines OQQ', ORR' be moved parallel to themselves until they become the tangents at P and P', we shall then obtain, if these tangents intersect in T,

$$TP^2 : TP'^2 :: SP : SP',$$

a result previously obtained (Art. 37).

49. Prop. XX. *If a circle intersect a parabola in four points, the two straight lines constituting any one of the three pairs of the chords of intersection are equally inclined to the axis.*

Taking the figure of the previous Article, let Q, Q', R, R' be the four points of intersection;

then $$OQ . OQ' = OR . OR',$$

and therefore SP, SP' are equal.

But, if SP, SP' be equal, the points P, P' are on opposite sides of, and are equidistant from the axis, and the tangents at P and P' are therefore equally inclined to the axis.

Hence the chords QQ', RR', which are parallel to these tangents, are equally inclined to the axis.

In the same manner it may be shewn that QR, $Q'R'$ are equally inclined to the axis, as also QR', $Q'R$.

50. Conversely, if two chords QQ', RR', which are not parallel, make equal angles with the axis, a circle can be drawn through $QQ'R'R$.

For, if the chords intersect in O, and OE be drawn parallel to the axis and meeting the curve in E, it may be shewn as above that

$$OQ . OQ' = 4SP . OE \text{ and } OR . OR' = 4SP' . OE,$$

P and P' being the vertices of the diameters bisecting the chords.

But the tangents at P and P', which are parallel to the chords, are equally inclined to the axis, and therefore SP is equal to SP'.

Hence $$OQ \,.\, OQ' = OR \,.\, OR',$$

and therefore a circle can be drawn through the points Q, Q', R, R'.

If the two chords are both perpendicular to the axis, it is obvious that a circle can be drawn through their extremities, and this is the only case in which a circle can be drawn through the extremities of parallel chords.

EXAMPLES.

1. FIND the locus of the centre of a circle which passes through a given point and touches a given straight line.

2. Draw a tangent to a parabola, making a given angle with the axis.

3. If the tangent at P meet the tangent at the vertex in Y,
$$AY^2 = AS \,.\, AN.$$

4. If the normal at P meet the axis in G, the focus is equidistant from the tangent at P and the straight line through G parallel to the tangent.

5. Given the focus, the position of the axis, and a tangent, construct the parabola.

6. Find the locus of the centre of a circle which touches a given straight line and a given circle.

7. Construct a parabola which has a given focus, and two given tangents.

8. The distance of any point on a parabola from the focus is equal to the length of the ordinate at that point produced to meet the tangent at the end of the latus rectum.

9. PT being the tangent at P, meeting the axis in T, and PN the ordinate, prove that $TY \,.\, TP = TS \,.\, TN$.

10. If SE be the perpendicular from the focus on the normal at P, shew that
$$SE^2 = AN \,.\, SP.$$

11. The locus of the vertices of all parabolas, which have a common focus and a common tangent, is a circle.

12. Having given the focus, the length of the latus rectum, and a tangent, construct the parabola.

13. If PSP' be a focal chord, and PN, $P'N'$ the ordinates, shew that
$$AN \,.\, AN' = AS^2.$$

Shew also that the latus rectum is a mean proportional between the double ordinates.

14. The locus of the middle points of the focal chords of a parabola is another parabola.

15. Shew that in general two parabolas can be drawn having a given straight line for directrix, and passing through two given points on the same side of the line.

16. Pp is a chord perpendicular to the axis, and the perpendicular from p on the tangent at P meets the diameter through P in R; prove that RP is equal to the latus rectum, and find the locus of R.

17. Having given the focus, describe a parabola passing through two given points.

18. The circle on any focal distance as diameter touches the tangent at the vertex.

19. The circle on any focal chord as diameter touches the directrix.

20. A point moves so that its shortest distance from a given circle is equal to its distance from a given diameter of the circle; prove that the locus is a parabola, the focus of which coincides with the centre of the circle.

21. Find the locus of a point which moves so that its shortest distance from a given circle is equal to its distance from a given straight line.

22. If APC be a sector of a circle, of which the radius CA is fixed, and a circle be described, touching the radii CA, CP, and the arc AP, the locus of the centre of this circle is a parabola.

23. If from the focus S of a parabola, SY, SZ be perpendiculars drawn to the tangent and normal at any point, YZ is parallel to the diameter.

24. Prove that the locus of the foot of the perpendicular from the focus on the normal is a parabola.

25. If PG be the normal, and GL the perpendicular from G upon SP, prove that GL is equal to the ordinate PN.

26. Given the focus, a point P on the curve, and the length of the perpendicular from the focus on the tangent at P, find the vertex.

27. A circle is described on the latus rectum as diameter,

and a common tangent QP is drawn to it and the parabola: shew that SP, SQ make equal angles with the latus rectum.

28. G is the foot of the normal at a point P of the parabola, Q is the middle point of SG, and X is the foot of the directrix: prove that

$$QX^2 - QP^2 = 4AS^2.$$

29. If PG the normal at P meet the axis in G, and if PF, PH, lines equally inclined to PG, meet the axis in F and H, the length SG is a mean proportional between SF and SH.

30. A triangle ABC circumscribes a parabola whose focus is S, and through A, B, C, lines are drawn respectively perpendicular to SA, SB, SC; shew that these pass through one point.

31. If PQ be the normal at P meeting the curve in Q, and if the chord PR be drawn so that PR, PQ are equally inclined to the axis, PRQ is a right angle.

32. A chord POQ meets a diameter, vertex A, in the point O, and PN, QM are the ordinates of P and Q referred to that diameter; prove that

$$AN.AM = AO^2.$$

33. Having given two points of a parabola, the direction of the axis, and the tangent at one of the points, construct the parabola.

34. Having given the vertex of a diameter, and a corresponding double ordinate, construct the parabola.

35. PM is an ordinate of a point P; a straight line parallel to the axis bisects PM, and meets the curve in Q; MQ meets the tangent at the vertex in T; prove that $3AT = 2PM$.

36. AB, CD are two parallel straight lines given in position, and AC is perpendicular to both, A and C being given points; in CD any point Q is taken, and in AQ, produced if necessary, a point P is taken, such that the distance of P from AB is equal to CQ; prove that the locus of P is a parabola.

37. If the tangent and normal at a point P of a parabola meet the tangent at the vertex (A) in K and L respectively, prove that

$$KL^2 : SP^2 :: SP - AS : AS,$$

S being the focus.

38. Having given the length of a focal chord, find its position.

39. If the ordinate of a point P bisects the subnormal of a point P', prove that the ordinate of P is equal to the normal of P'.

40. A parabola being traced on a plane, find its axis and vertex.

41. If PV, $P'V'$ be two diameters, and PV', $P'V$ ordinates to these diameters,

$$PV = P'V'.$$

42. If one side of a triangle be parallel to the axis of a parabola, the other sides will be in the ratio of the tangents parallel to them.

43. If PSp, QSq be focal chords,

$$PS . Sp : QS . Sq :: Pp : Qq.$$

44. QVQ' is an ordinate of a diameter PV, and any chord PR meets QQ' in N, and the diameter through Q in L; prove that

$$PL^2 = PN . PR.$$

45. Describe a parabola passing through three given points, and having its axis parallel to a given line.

46. If AP, AQ be two chords drawn from the vertex at right angles to each other, and PN, QM be ordinates, the latus rectum is a mean proportional between AN and AM.

47. PSp is a focal chord of a parabola; prove that AP, Ap meet the latus rectum in two points whose distances from the focus are equal to the ordinates of p and P respectively.

48. A chord PQ of a parabola is normal to the parabola at P, and the angle PSQ is a right angle; shew that $SQ = 2SP$.

49. From any point Q in the line BQ which is perpendicular to the axis CAB of a parabola, vertex A, QR is drawn parallel to the axis to meet the curve in R; prove that if CA be equal to AB, the lines AQ, CR will meet on the parabola.

50. From the vertex of a parabola a perpendicular is drawn on the tangent at any point; prove that the locus of its intersection with the diameter through the point is a straight line.

51. If two tangents to a parabola be drawn from any point in its axis, and if any other tangent intersect these two in P and Q, prove that $SP = SQ$.

52. T is a point on the tangent at P, such that the perpendicular from T on SP is of constant length; prove that the locus of T is a parabola.

If the constant length be $2AS$, prove that the vertex of the locus is on the directrix.

53. The area of the triangle formed by three tangents to a parabola is equal to one half that of the triangle formed by joining the points of contact.

54. If the normal at a point P of a parabola meet the curve in Q, and the tangents at P and Q intersect in T, prove that T and P are equidistant from the directrix.

55. Two tangents TP, TQ are drawn to a parabola cutting the directrix in P', Q'. If the chord PQ cut the directrix in R', and if S be the focus, prove that $SP' : SQ' :: RP' : RQ'$.

56. If two equal tangents to a parabola be cut by a third tangent, the alternate segments of the two tangents will be equal.

57. If AP be a chord through the vertex, and if PL, perpendicular to AP, and PG, the normal at P, meet the axis in L, G respectively, $GL =$ half the latus rectum.

58. If PSQ be a focal chord, A the vertex, and PA, QA be produced to meet the directrix in P', Q' respectively, then $P'SQ'$ will be a right angle.

59. The tangents at P and Q intersect in T, and the tangent at R intersects TP and TQ in C and D; prove that

$$PC : CT :: CR : RD :: TD : DQ.$$

60. From any point D in the latus rectum of a parabola, a straight line DP is drawn, parallel to the axis, to meet the curve in P; if X be the foot of the directrix, and A the vertex, prove that AD, XP intersect in the parabola.

61. PSp is a focal chord, and upon PS and pS as diameters circles are described; prove that the length of either of their common tangents is a mean proportional between AS and Pp.

62. If AQ be a chord of a parabola through the vertex A, and QR be drawn perpendicular to AQ to meet the axis in R; prove that AR will be equal to the chord through the focus parallel to AQ.

63. If from any point P of a circle, PC be drawn to the centre C, and a chord PQ be drawn parallel to the diameter AB, and bisected in R; shew that the locus of the intersection of CP and AR is a parabola.

64. A circle, the diameter of which is three-fourths of the latus rectum, is described about the vertex A of a parabola as centre; prove that the common chord bisects AS.

65. Shew that straight lines drawn perpendicular to the tangents of a parabola through the points where they meet a given fixed line perpendicular to the axis are in general tangents to a confocal parabola.

66. If QR be a double ordinate, and PD a straight line drawn parallel to the axis from any point P of the curve, and meeting QR in D, prove, from Art. 25, that

$$QD \, . \, RD = 4AS \, . \, PD.$$

67. Prove, by help of the preceding theorem, that, if QQ' be a chord parallel to the tangent at P, QQ' is bisected by PD, and hence determine the locus of the middle point of a series of parallel chords.

68. If from either end of a double ordinate QVQ' to any diameter PV, a perpendicular QD be let fall upon the diameter, prove, by help of problem (66), and of Art. 32, that

$$QD^2 = 4AS \, . \, PV.$$

69. Find the locus of the foci of the parabolas which have a common vertex and a common tangent.

70. From the points where the normals to a parabola meet the axis, lines are drawn perpendicular to the normals: shew that these lines will be tangents to an equal parabola.

71. Inscribe in a given parabola a triangle having its sides parallel to three given straight lines.

72. PNP' is a double ordinate, and through a point of the parabola RQL is drawn perpendicular to PP' and meeting PA, or PA produced in R; prove that

$$PN : NL :: LR : RQ.$$

73. PNP' is a double ordinate, and through R, a point in the tangent at P, RQM is drawn perpendicular to PP' and meeting the curve in Q; prove that

$$QM : QR :: P'M : PM.$$

74. If from the point of contact of a tangent to a parabola, a chord be drawn, and a line parallel to the axis meeting the chord, the tangent, and the curve, shew that this line will be divided by them in the same ratio as it divides the chord.

75. PSp is a focal chord of a parabola, RD is the directrix meeting the axis in D, Q is any point in the curve; prove that if QP, Qp produced meet the directrix in R, r, half the latus rectum will be a mean proportional between DR and Dr.

76. A chord of a parabola is drawn parallel to a given straight line, and on this chord as diameter a circle is described; prove that the distance between the middle points of this chord, and of the chord joining the other two points of intersection of the circle and parabola, will be of constant length.

77. If a circle and a parabola have a common tangent at P, and intersect in Q and R; and if QV, UR be drawn parallel to the axis of the parabola meeting the circle in V and U respectively, then will VU be parallel to the tangent at P.

78. If PV be the diameter through any point P, QV a semi-ordinate, Q' another point in the curve, and $Q'P$ cut QV in R, and $Q'R'$ the diameter through Q' meet QV in R', then

$$VR \,.\, VR' = QV^2.$$

79. PQ, PR are any two chords; PQ meets the diameter through R in the point F, and PR meets the diameter through Q in E; prove that EF is parallel to the tangent at P.

80. If parallel chords be intersected by a diameter, the distances of the points of intersection from the vertex of the diameter are in the ratio of the rectangles contained by the segments of the chords.

81. If tangents be drawn to a parabola from any point P in the latus rectum, and if Q, Q' be the point of contact, the semi-latus rectum is a geometric mean between the ordinates of Q and Q', and the distance of P from the axis is an arithmetic mean between the same ordinates.

82. If A', B', C' be the middle points of the sides of a triangle ABC, and a parabola drawn through A', B', C' meet the sides again in A'', B'', C'', then will the lines AA'', BB'', CC'' be parallel to each other.

83. A circle passing through the focus cuts the parabola in two points. Prove that the angle between the tangents to the circle at those points is four times the angle between the tangents to the parabola at the same points.

84. The locus of the points of intersection of normals at the extremities of focal chords of a parabola is another parabola.

85. Having given the vertex, a tangent, and its point of contact, construct the parabola.

86. *PSp* is a focal chord of a parabola; shew that the distance of the point of intersection of the normals at *P* and *p* from the directrix varies as the rectangle contained by *PS*, *pS*.

87. *TP*, *TQ* are tangents to a parabola at *P* and *Q*, and *O* is the centre of the circle circumscribing *PTQ*; prove that *TSO* is a right angle.

88. *P* is any point of a parabola whose vertex is *A*, and through the focus *S* the chord *QSQ'* is drawn parallel to *AP*; *PN*, *QM*, *Q'M'*, being perpendicular to the axis, shew that *SM* is a mean proportional between *AM*, *AN*, and that

$$MM' = AP.$$

89. If a circle cut a parabola in four points, two on one side of the axis, and two on the other, the sum of the ordinates of the first two is equal to the sum of the ordinates of the other two points.

Extend this theorem to the case in which three of the points are on one side of the axis and one on the other.

90. The tangents at *P* and *Q* meet in *T*, and *TL* is the perpendicular from *T* on the axis; prove that if *PN*, *QM* be the ordinates of *P* and *Q*,

$$PN . QM = 4AS . AL.$$

91. The tangents at *P* and *Q* meet in *T*, and the lines *TA*, *PA*, *QA*, meet the directrix in *t*, *p*, and *q*: prove that

$$tp = tq.$$

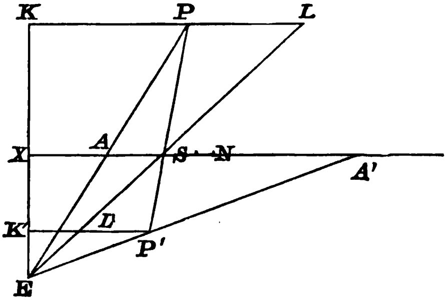
CHAPTER III.

THE ELLIPSE.

Def. *An ellipse is the curve traced out by a point which moves in such a manner that its distance from a given point is in a constant ratio of less inequality to its distance from a given straight line.*

Tracing the Curve.

51. Let S be the focus, EX the directrix, and SX the perpendicular on EX from S.



Divide SX at the point A in the given ratio; the point A is the vertex.

From any point E in EX, draw EAP, ESL, and through S draw SP making the angle PSL equal to LSN, and meeting EAP in P.

Through P draw LPK perpendicular to the directrix and meeting ESL in L.

Then the angle $PSL = LSN = SLP$,

$$\therefore SP = PL.$$

Also $PL : PK :: SA : AX.$

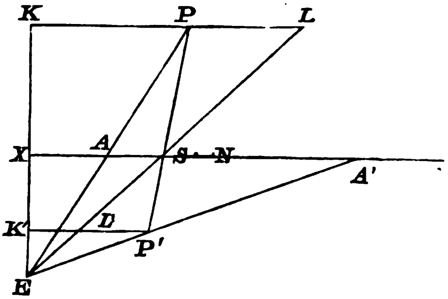
Hence $SP : PK :: SA : AX,$

and P is therefore a point in the curve.

Again, in the axis XAN find a point A' such that

$$SA' : A'X :: SA : AX;$$

this point is evidently on the same side of the directrix as the point A, and is another vertex of the curve.



Join EA' meeting PS produced in P', and draw $P'L'K'$ perpendicular to the directrix and meeting ES in L'.

Then $P'L' : P'K' :: SA' : A'X$

$:: SA : AX,$

and the angle $SL'P' = L'SA = L'SP'$;

$\therefore P'L' = SP'.$

Hence P' is also a point in the curve, and PSP' is a focal chord.

By giving E a series of positions on the directrix we shall obtain a series of focal chords, and we can also, as in Art. (3), find other points of the curve lying in the lines KP, $K'P'$, or in these lines produced.

We can thus find any number of points in the curve.

52. Def. *The distance* AA′ *is the major axis.*

The middle point C *of* AA′ *is called the centre of the ellipse.*

If through C *the double ordinate* BCB′ *be drawn,* BB′ *is called the minor axis.*

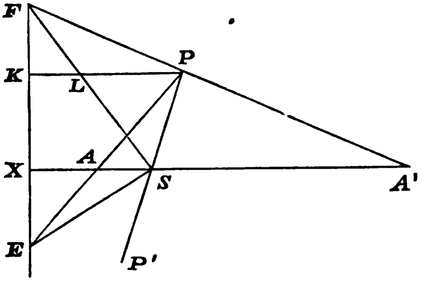
Any straight line drawn through the centre, and terminated by the curve, is called a diameter.

The lines ACA′, BCB′ *are called the principal diameters, or, briefly, the axes of the curve.*

The line ACA′ *is also sometimes called the transverse axis, and* BCB′ *the conjugate axis.*

53. Prop. I. *If* P *be any point of an ellipse, and* AA′ *the axis major, and if* PA, A′P, *when produced, meet the directrix in* E *and* F, *the distance* EF *subtends a right angle at the focus.*

Draw PLK perpendicular to the directrix, meeting SF in L, and the directrix in K.



Then $$PL : PK :: SA' : A'X$$
$$:: SP : PK;$$
$$\therefore PL = SP,$$

and the angle $LSP = PLS = LSX$,
that is, FS bisects the angle ASP.

But, if PS be produced to P', ES bisects the angle ASP';

$\therefore$ ESF is a right angle.

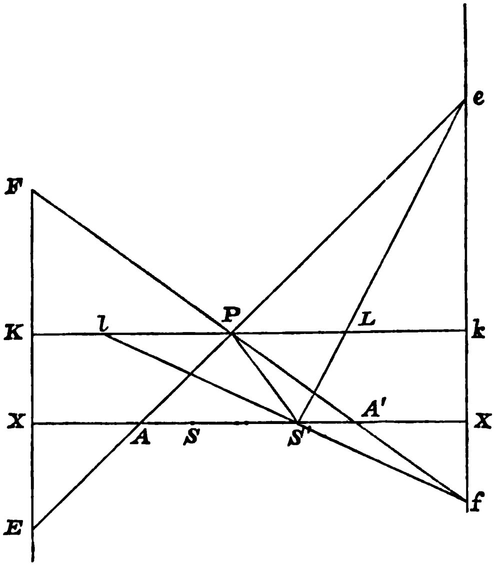
54. By help of the preceding theorem we shall now prove the existence of another focus and directrix corresponding to the vertex A'.

In AA' produced take a point X' such that $A'X'=AX$, and in AA' take a point S' such that $A'S'=AS$.

Through X' draw a straight line $eX'f$ perpendicular to the axis and let EP, FP produced meet this line in e and f. Join eS', and fS'.

$$\begin{aligned} \text{Then} \qquad eX' : EX &:: AX' : AX \\ &:: A'X : A'X' \\ &:: FX : fX'; \end{aligned}$$

$$\therefore\ eX' . fX' = EX . FX = SX^2 = S'X'^2.$$



Hence $eS'f$ is a right angle.

Through P draw KPk parallel to the axis, meeting eS' and fS' produced in L and l.

Then $PL : Pk :: S'A : AX'$,

and $Pl : Pk :: S'A' : A'X'$;

$$\therefore PL = Pl.$$

Moreover, $LS'l$ being a right angle,

$$S'P = Pl,$$

$$\therefore S'P : Pk :: S'A' : A'X',$$

and the curve can be described by means of the focus S' and the directrix eX'.

Hence also it follows that the curve is symmetrical with regard to BCB', and that it lies wholly between the tangents at A and A'.

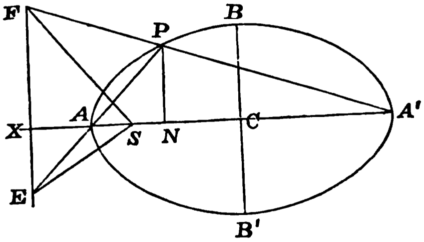
If SA be equal to AX, the point A', and therefore the points S' and X' will be at an infinite distance from S and A.

Hence a parabola is the limiting form of an ellipse, the axis major of which is indefinitely increased in magnitude, while the distance SA remains finite.

55. Prop. II. *If PN be the ordinate of any point P of an ellipse, ACA' the axis major, and BCB' the axis minor,*

$$PN^2 : AN.NA' :: BC^2 : AC^2.$$

Join PA, $A'P$, and let these lines produced meet the directrix in E and F.



Then, $PN : AN :: EX : AX$,

and $PN : A'N :: FX : A'X$;

$$\therefore PN^2 : AN.NA' :: EX.FX : AX.A'X$$

$$:: SX^2 : AX.A'X,$$

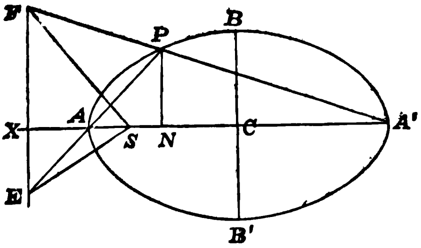
since ESF is a right angle (Prop. I.), that is, PN^2 is to $AN.NA'$ in a constant ratio.

Hence, taking PN coincident with BC, in which case

$$AN = NA' = AC,$$

$$BC^2 : AC^2 :: SX^2 : AX.A'X,$$

and $$\therefore PN^2 : AN.NA' :: BC^2 : AC^2.$$



This may be also written

$$PN^2 : AC^2 - CN^2 :: BC^2 : AC^2.$$

Cor. If PM be the perpendicular from P on the axis minor,

$$CM = PN, \quad PM = CN,$$

and $$CM^2 : AC^2 - PM^2 :: BC^2 : AC^2.$$

Hence $$AC^2 : AC^2 - PM^2 :: BC^2 : CM^2,$$

and $$\therefore AC^2 : PM^2 :: BC^2 : BC^2 - CM^2,$$

or $$PM^2 : BM.MB' :: AC^2 : BC^2$$

56. Conversely, if a point P move in such a manner that PN^2 is to $AN.NA'$ in a constant ratio, PN being the distance of P from the line joining two fixed points A, A', and N being between A and A', the locus of P will be an ellipse of which AA' is the axis.

For, taking C as the middle point of AA', draw BC at right angles to AA', and such that BC^2 is to AC^2 in the given constant ratio; then, if any point N be taken in AA, the corresponding point of the locus evidently coincides with a point of the ellipse of which AC and BC are the semi-axes.

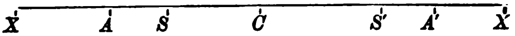
It will be shewn in the Appendix that the definition of an ellipse is a property directly deducible from the relation

$$PN^2 : AN.NA' :: BC^2 : AC^2.$$

57. PROP. III. *If ACA' be the axis major, C the centre, S one of the foci, and X the foot of the directrix,*

$$CS : CA :: CA : CX :: SA : AX,$$

and $$CS : CX :: CS^2 : CA^2.$$



For $$S'A : SA :: AX' : AX$$
$$:: A'X : AX;$$
$$\therefore SS' : SA :: AA' : AX,$$
or $$CS : CA :: SA : AX.$$

Again, $$SA' : SA :: AX' : AX;$$
$$\therefore AA' : SA :: XX' : AX,$$
or $$CA : CX :: SA : AX;$$
$$\therefore CS : CA :: CA : CX,$$
or $$CS.CX = CA^2.$$

Also $$CS : CX :: CS^2 : CS.CX$$
$$:: CS^2 : CA^2.$$

58. **Prop. IV.** *If S be a focus, and B an extremity of the axis minor,*

$$SB = AC \text{ and } BC^2 = AS \cdot SA'.$$

For, joining SB in the figure of Art. 55,

$$\begin{aligned} SB : CX &:: SA : AX \\ &:: CA : CX, \end{aligned}$$

by the previous Article,

$$\therefore SB = CA.$$

Also $$\begin{aligned} BC^2 = SB^2 - SC^2 &= AC^2 - SC^2 \\ &= AS \cdot SA'. \end{aligned}$$

59. **Prop. V.** *The semi-latus rectum SR is a third proportional to AC and BC.*

For, Prop. II.,

$$SR^2 : AS \cdot SA' :: BC^2 : AC^2;$$

$$\therefore SR^2 : BC^2 :: BC^2 : AC^2,$$

or $$SR : BC :: BC : AC.$$

Cor. Since $$\begin{aligned} SR : SX &:: SA : AX \\ &:: SC : AC, \end{aligned}$$

it follows that $SX \cdot SC = SR \cdot AC = BC^2$;

and hence also that

$$SX : CX :: BC^2 : AC^2.$$

60. **Prop. VI.** *The sum of the focal distances of any point is equal to the axis major.*

Let PN be the ordinate of a point P (Fig. Art. 54), then

$$S'P : SP :: NX' : NX;$$

$$\therefore S'P + SP : SP :: XX' : NX,$$

or $$\begin{aligned} S'P + SP : XX' &:: SP : NX \\ &:: SA : AX \\ &:: AA' : XX'; \end{aligned}$$

$$\therefore S'P + SP = AA'.$$

Cor. Since $SP : NX :: SA : AX$

$:: AC : CX;$

$\therefore AC : SP :: CX : NX,$

$AC - SP : SP :: CN : NX,$

and $AC - SP : CN :: SA : AX.$

Also, $AC - SP = S'P - AC;$

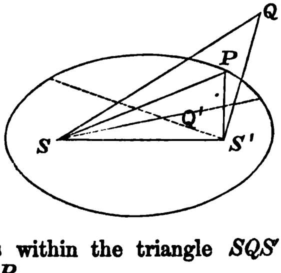
$\therefore S'P - AC : CN :: SA : AX.$

Mechanical Construction of the Ellipse.

61. Fasten the ends of a piece of thread to two pins fixed on a board, and trace a curve on the board with a pencil pressed against the thread so as to keep it stretched; the curve traced out will be an ellipse, having its foci at the points where the pins are fixed, and having its major axis equal to the length of the thread.

62. Prop. VII. *The sum of the distances of a point from the foci of an ellipse is greater or less than the major axis according as the point is outside or inside the ellipse.*

If the point be without the ellipse join SQ, $S'Q$, and take a point P on the intercepted arc of the curve.



Then P is within the triangle SQS' and therefore, joining SP, $S'P$,

$$SQ + S'Q > SP + S'P, \text{ Euclid I. 21},$$

i.e. $$SQ + S'Q > AA'.$$

If Q be within the ellipse, let SQ, $S'Q$ produced meet the curve and take a point P on the intercepted arc.

Then Q is within the triangle SPS', and

$$\therefore SP+S'P>SQ+S'Q,$$

i.e. $$SQ+S'Q<AA'.$$

63. DEF. *The circle described on the axis major as diameter is called the auxiliary circle.*

PROP. VIII. *If the ordinate NP of an ellipse be produced to meet the auxiliary circle in Q,*

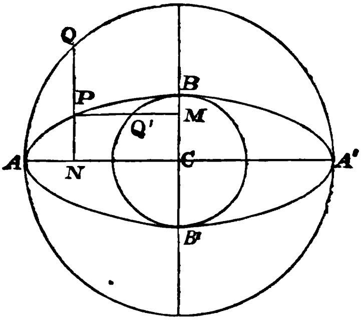
$$PN : QN :: BC : AC.$$

For, Art. 55,

$$PN^2 : AN.NA' :: BC^2 : AC^2,$$

and, by a property of the circle,

$$QN^2=AN.NA';$$



$$\therefore PN : QN :: BC : AC.$$

COR. Similarly, if PM the perpendicular on BB' meet in Q' the circle described on BB' as diameter

$$PM : Q'M :: AC : BC.$$

For $$PM^2 : BM.MB' :: AC^2 : BC^2,$$

and $$BM.MB'=Q'M^2.$$

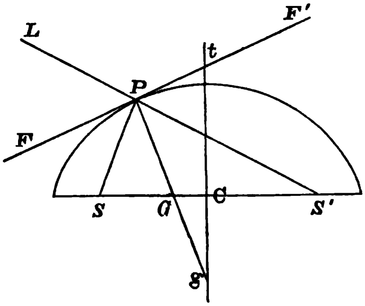
Properties of the Tangent and Normal.

64. Prop. IX. *The normal at any point bisects the angle between the focal distances of that point, and the tangent is equally inclined to the focal distances.*

Let the normal at P meet the axis in G; then, Art. 15

$$SG : SP :: SA : AX,$$

and
$$S'G : S'P :: SA : AX.$$



Hence
$$SG : SG' :: SP : S'P,$$

and therefore the angle SPS' is bisected by PG.

Also FPF' being the tangent, and GPF, GPF' being right angles, it follows that the angles SPF, $S'PF'$ are equal, or that the tangent is equally inclined to the focal distances.

Hence if $S'P$ be produced to L, the tangent bisects the angle SPL.

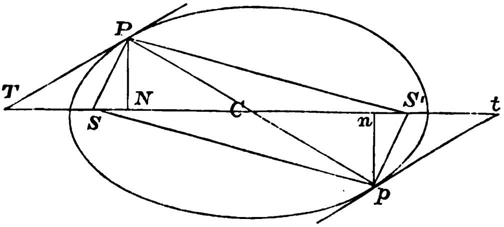
Cor. If a circle be described about the triangle SPS', its centre will lie in BCB', which bisects SS' at right angles; and since the angles SPG, $S'PG$ are equal, and equal angles stand upon equal arcs, the point g, in which PG produced meets the minor axis, is a point in the circle.

Also, if the tangent meet the minor axis in t, the point t is on the same circle, since gPt is a right angle.

Hence, *Any point P of an ellipse, the two foci, and the points of intersection of the tangent and normal at P with the minor axis lie on the same circle.*

65. Prop. X. *Every diameter is bisected at the centre and the tangents at the ends of a diameter are parallel.*

Let PCp be a diameter, PN, pN the ordinates of P and p.



Then $CN^2 : Cn^2 :: PN^2 : pn^2$

$$:: AC^2 - CN^2 : AC^2 - Cn^2, \text{ Art. } 55;$$

$$\therefore CN^2 : AC^2 :: Cn^2 : AC^2.$$

Hence $CN = Cn$ and $\therefore CP = Cp$.

Draw the focal distances; then, since Pp and SS' bisect each other in C, the figure $SPS'p$ is a parallelogram, and the angle

$$SPS' = SpS'.$$

But the tangents PT, pt are equally inclined to the focal distances;

$$\therefore \text{ the angle } SPT = S'pt,$$

and, adding the equal angles CPS, CpS',

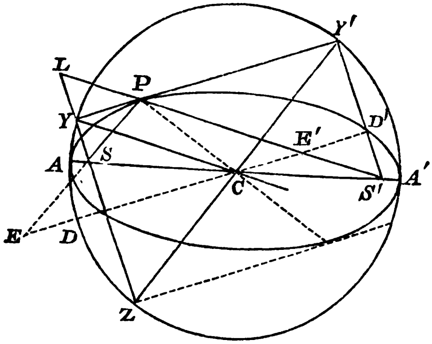
$$CPT = Cpt;$$

$\therefore$ PT and pt are parallel.

COR. Since Sp and $S'p$ are equally inclined to the tangent at p, it follows that SP and Sp make equal angles with the tangents at P and p.

66. PROP. XI. *The perpendiculars from the foci on any tangent meet the tangent on the auxiliary circle, and the semi-minor axis is a mean proportional between their lengths.*

Let SY, $S'Y'$ be the perpendiculars; join $S'P$, and let SY, $S'P$ produced meet in L.



The angles SPY, YPL being equal, and PY being common, the triangles are equal in all respects;

$$\therefore PL = SP,\ SY = YP,$$

and $$S'L = S'P + PL = S'P + SP = AA'.$$

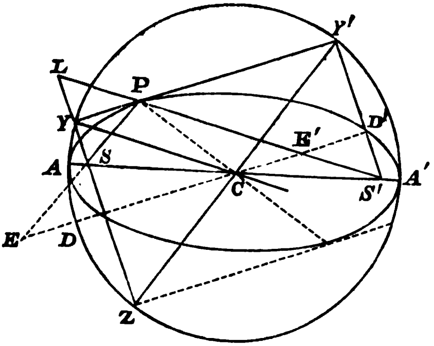
Join CY, then C being the middle point of SS', and Y of SL, CY is parallel to $S'L$,

and $$\therefore S'L = 2CY.$$

Hence $CY = AC$, and Y is a point on the auxiliary circle.

Similarly by producing SP, $S'Y'$ it may be shewn that Y' is also on the auxiliary circle.

Let YS produced meet the circle in Z, and join $Y'Z$; then $Y'YZ$ being a right angle, $Y'Z$ is a diameter and passes through C.



Hence the triangles SCZ, $S'CY'$ are equal, and

$$SY.S'Y' = SY.SZ = AS.SA' = BC^2.$$

Cor. (1). If P' be the other extremity of the diameter through P, the tangent at P' is parallel to PY, and therefore Z is the foot of the perpendicular from S on the tangent at P'.

Cor. (2). If the diameter DCD', drawn parallel to the tangent at P, meet SP, $S'P$ in E and E', $PECY'$ is a parallelogram, for CY' is parallel to SP, and CE to PY';

$$\therefore PE = CY' = AC;$$

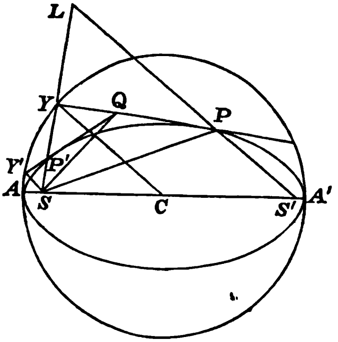
and similarly $\qquad PE' = CY = AC.$

67. Prop. XII. *To draw tangents from a given point to an ellipse.*

For this purpose we may employ the general construction of Art. (14), or the following.

Let Q be the given point; upon SQ as diameter describe a circle cutting the auxiliary circle in Y and Y'; YQ and $Y'Q$ will be the required tangents.

Producing SY to L so that $YL = SY$, join $S'L$ cutting the line YQ in P.



The triangles SPY, LPY are equal in all respects, since $SY = YL$ and PY is common and perpendicular to SL;

$$\therefore\ SP = PL \text{ and } S'L = S'P + PL = S'P + SP;$$

but, joining CY, $S'L = 2CY = 2AC$;

$$\therefore\ SP + S'P = 2AC,$$

and P is therefore a point on the ellipse.

Also the angle $SPY = YPL$,

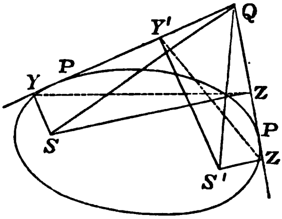
and $\therefore$ QP is the tangent at P.

A similar construction will give the point of contact of the other tangent QP'.

Referring to Art. 31, it will be seen that the construction is the same as that given for the parabola, the ultimate form of the circle being, for the parabola, the tangent at the vertex.

68. Prop. XIII. *If two tangents be drawn to an ellipse from an external point, they are equally inclined to the focal distances of that point.*

Let QP, QP' be the tangents, SY, $S'Y'$, SZ, $S'Z'$ the



perpendiculars from the foci on the tangents; join YZ, $Y'Z'$.

Then, Art. 66, $SY.S'Y' = SZ.S'Z'$;

$$\therefore SY : SZ :: S'Z' : S'Y'.$$

A circle can be drawn through the points $SYQZ$, since SYQ, SZQ are right angles; and YSZ and YQZ are equal to two right angles, as are also $Y'S'Z'$ and $Y'QZ'$; therefore the angle $YSZ = Y'S'Z'$, and the triangles YSZ, $Y'S'Z'$ are similar.

Hence the angle $SZY = S'Y'Z'$.

But $SZY = SQY$ in the same segment, and similarly

$$S'Y'Z' = S'QZ';$$

therefore the angle $SQP = S'QP'$.

69. Def. *Ellipses which have the same foci are called confocal ellipses.*

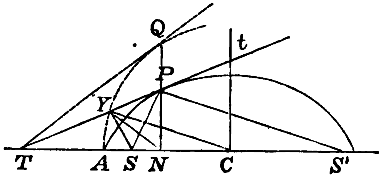
If Q be a point in a confocal ellipse the normal at Q bisects the angle SQS' and therefore bisects the angle PQP'.

Hence, *If from any point of an ellipse tangents are drawn to a confocal ellipse, these tangents are equally inclined to the normal at the point.*

By reference to the remark of Art. 40, it will be seen that this theorem includes that of Art. 40 as a particular case.

70. Prop. XIV. *If PT the tangent at P meet the axis major in T, and PN be the ordinate,*

$$CN.CT=AC^2.$$



Draw the focal distances SP, $S'P$, and the perpendicular SY on the tangent, and join NY, CY.

Then, as in Art. 66, CY is parallel to $S'P$; therefore the angle

$$CYP=S'Pt=SPY$$
$$=SNY,$$

since a circle can be drawn through the four points $SYPN$.

Hence $$CYT=CNY,$$

and the triangles CYT, CNY are equiangular.

Therefore $$CN : CY :: CY : CT$$

or $$CN.CT=CY^2=AC^2.$$

Cor. (1). $CN.NT=CN.CT-CN^2=AC^2-CN^2$

$$=AN.NA'.$$

Cor. (2). Hence it follows that *tangents at the extremities of a common ordinate of an ellipse and its auxiliary circle meet the axis in the same point.*

For, if NP produced meet the auxiliary circle in Q, and the tangent at Q meet the axis in T',

$$CN.CT'=CQ^2=AC^2,$$

therefore T' coincides with T.

And more generally it is evident that, *If any number of ellipses be described having the same major axis, and an ordinate be drawn cutting the ellipses, the tangents at the points of section will all meet the common axis in the same point.*

71. Prop. XV. *If the tangent at P meet the axis minor in t, and PN be the ordinate,*

$$Ct.PN=BC^2.$$

For, $Ct : PN :: CT : NT$, Fig. Art. 70.

$$\because\ Ct.PN : PN^2 :: CT.CN : CN.NT$$

$$:: AC^2 : AN.NA', \text{ Cor. 1, Art. 70,}$$

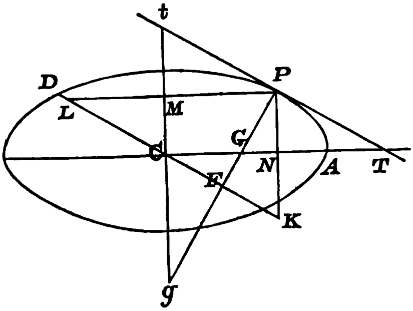
$$:: BC^2 : PN^2.$$

$$\therefore\ Ct.PN=BC^2.$$

72. Prop. XVI. *If the normal at P meet the axes in G and g, and the diameter parallel to the tangent at P in F,*

$$PF.FG=BC^2, \text{ and } PF.Pg=AC^2.$$

Let PN, PM, perpendiculars on the axes, meet the diameter in K and L, and let the tangent at P meet the axes in T and t.



Then, since a circle can be drawn through $GFKN$,

$$PF.PG=PN.PK=PN.Ct=BC^2.$$

Similarly, since a circle can be drawn through $LMFg$,

$$PF.Pg = PM.PL = CN.CT = AC^2.$$

COR. Since PGN, PgM are similar triangles,

$$NG : PM :: PG : Pg$$
$$:: PF.PG : PF.Pg$$
$$:: BC^2 : AC^2,$$

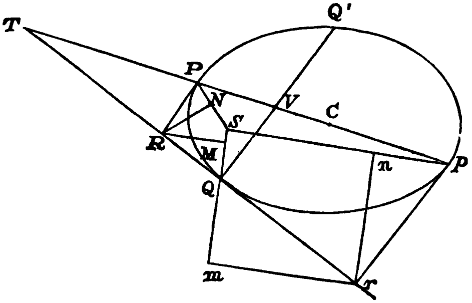
or $$NG : CN :: BC^2 : AC^2.$$

Hence also

$$CG : CN :: AC^2 - BC^2 : AC^2$$
$$:: SC^2 : AC^2.$$

73. PROP. XVII. *If PCp be a diameter, and QVQ' a chord parallel to the tangent at P and meeting Pp in V, and if the tangent at Q meet pP produced in T,*

$$CV.CT = CP^2.$$



Let TQ meet the tangents at P and p in R and r, and S being a focus, join SP, SQ, Sp.

Let fall perpendiculars RN, RM, rn, rm upon these focal distances;

then, since the angle $SPR = Spr$, Cor. Art. (65),

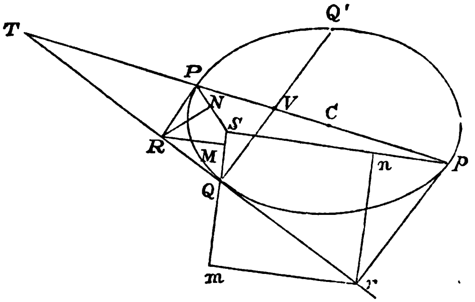
$$RP : rp :: RN : rn$$

$$:: RM : rm, \text{ Art. 13,}$$

$$:: RQ : rQ;$$

but $$TR : Tr :: RP : rp.$$

$$\therefore TR : Tr :: RQ : rQ.$$



Hence $$TP : Tp :: PV : Vp,$$

or $$CT - CP : CT + CP :: CP - CV : CP + CV;$$

$$\therefore CT : CP :: CP : CV,$$

or $$CT \, . \, CV = CP^2.$$

Cor. 1. Hence, since CV and CP are the same for the point Q', the tangent at Q' passes through T.

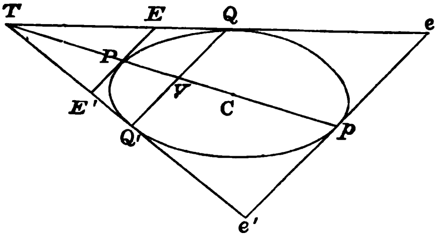
Cor. 2. Since $Tp : TP :: Tp - TV : TV - TP$, it follows that $TPVp$ is harmonically divided.

Properties of Conjugate Diameters.

74. Prop. XVIII. *A diameter bisects all chords parallel to the tangents at its extremities.*

We have shewn (Art. 16), that, if QQ' be a chord of a conic, TQ, TQ' the tangents at Q, Q', and EPE' a tangent parallel to QQ', the length EE' is bisected at P.

Draw the diameter PCp; the tangent epe' at p is parallel to EP, Art. (65), and is therefore parallel to QQ'.



Hence $ep=pe'$, and P, p being the middle points of the parallels ee', EE', the line Pp passes through T, and moreover bisects QQ'.

Similarly, if any other chord qq' be drawn parallel to QQ' the tangents at q and q' will meet in pP produced, and qq' will be bisected by pP.

Cor. Hence, if QQ', qq' be two chords parallel to the tangent at P, the chords Qq, $Q'q'$ will meet in CP or CP produced.

75. Def. *The diameter DCd, drawn parallel to the tangent at P, is said to be conjugate to PCp.*

A diameter therefore bisects all chords parallel to its conjugate.

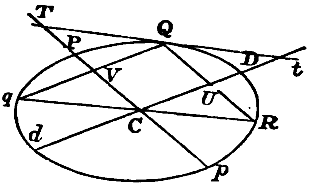
Prop. XIX. *If the diameter DCd be conjugate to PCp, then will PCp be conjugate to DCd.*

Let the chord QVq be parallel to DCd, and therefore bisected by PC, and draw the diameter qCR.

Join QR meeting CD in U;

then $RC = Cq$, and $QV = Vq$;

$\therefore$ QR is parallel to CP.



Also $QU : UR :: qC : CR$,

and therefore $QU = UR$.

That is, CD bisects the chords parallel to PCp; therefore PCp is conjugate to DCd.

DEF. *Chords drawn from the extremities of a diameter to any point of the ellipse are called supplemental chords.*

Thus qQ, RQ are supplemental chords, and hence it appears that supplemental chords are parallel to conjugate diameters.

DEF. *A line QV drawn from a point Q of an ellipse, parallel to the tangent at P and terminated by the diameter PCp, is called an ordinate of that diameter, and QVq is the double ordinate if QV produced meet the curve in q.*

76. PROP. XX. *If PCp, DCd be conjugate diameters, and QV an ordinate of Pp,*

$$QV^2 : PV.Vp :: CD^2 : CP^2.$$

Let the tangent at Q, fig. Art. 75, meet CP, CD produced in T and t, and draw QU parallel to CP and meeting CD in U.

Then $CP^2 = CV \cdot CT,$

and $CD^2 = CU \cdot Ct = QV \cdot Ct;$

$$\therefore CD^2 : CP^2 :: QV \cdot Ct : CV \cdot CT$$

$$:: QV^2 : CV \cdot VT,$$

and $CV \cdot VT = CV \cdot CT - CV^2 = CP^2 - CV^2$

$$= PV \cdot Vp,$$

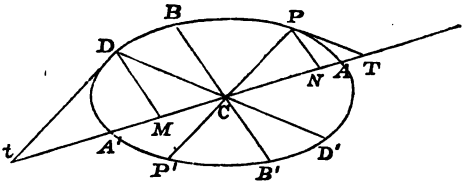
$$\therefore CD^2 : CP^2 :: QV^2 : PV \cdot Vp.$$

77. Prop. XXI. *If ACA', BCB' be a pair of conjugate diameters, PCP', DCD' another pair, and if PN, DM be ordinates of ACA',*

$$CN^2 = AM \cdot MA', \quad CM^2 = AN \cdot NA',$$

$$CM : PN :: AC : BC,$$

and $DM : CN :: BC : AC.$



Let the tangents at P and D meet ACA' in T and t.

Then $CN \cdot CT = AC^2 = CM \cdot Ct;$

hence $CM : CN :: CT : Ct$

$$:: PT : CD$$

$$:: PN : DM$$

$$:: CN : Mt,$$

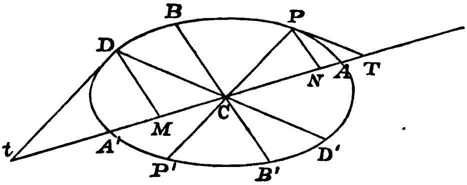
$$\therefore CN^2 = CM \cdot Mt = AC^2 - CM^2 = AM \cdot MA',$$

and similarly, $CM^2 = AN \cdot NA'.$

Also $DM^2 : AM . MA' :: BC^2 : AC^2$,

$$\therefore DM : CN :: BC : AC,$$

and similarly $CM : PN :: AC : BC$.



COR. We have shewn in the course of the proof that

$$CN^2 + CM^2 = AC^2.$$

By similar reasoning it appears that if Pn, Dm, be ordinates of BCB',

$$Cn^2 + Cm^2 = BC^2;$$

$$\therefore PN^2 + DM^2 = BC^2.$$

It should be noticed that these relations are shewn to be true when ACA', BCB' are any conjugate diameters including of course the principal axes.

78. PROP. XXII. *If CP, CD be conjugate semi-diameters, and AC, BC the principal semi-diameters,*

$$CP^2 + CD^2 = AC^2 + BC^2.$$

From the preceding article,

$$CN^2 + CM^2 = AC^2,$$

and $$PN^2 + DM^2 = BC^2;$$

also BCB being in this case a right angle,

$$PN^2 + CN^2 = CP^2,$$

and $$DM^2 + CM^2 = CD^2,$$

$$\therefore CP^2 + CD^2 = AC^2 + BC^2.$$

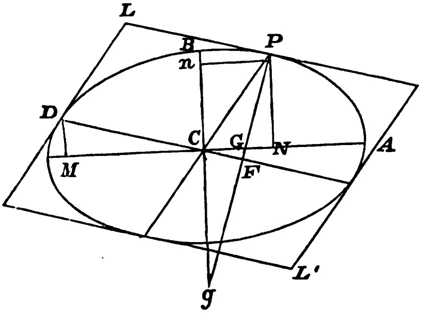
79. Prop. XXIII. *If the normal at P meet the principal axes in G and g,*

$$PG : CD :: BC : AC,$$

and $$Pg : CD :: AC : BC.$$

For, the triangles DCM, PGN being similar,

$$PG : CD :: PN : CM$$
$$:: BC : AC.$$



So also Pgn and DCM are similar, and

$$Pg : CD :: Pn : DM$$
$$:: AC : BC.$$

80. Prop. XXIV. *The parallelogram formed by the tangents at the ends of conjugate diameters is equal to the rectangle contained by the principal axes.*

For, taking the preceding figure,

$$PG : BC :: CD : AC;$$

but $PG : BC :: BC : PF$, Art. 72,

$$\therefore CD : AC :: BC : PF,$$

and $$CD . PF = AC . BC,$$

whence the theorem stated.

81. **Prop. XXV.** *If SP, $S'P$ be the focal distances of P, and CD be conjugate to CP,*

$$SP\,.\,S'P = CD^2.$$

Let CD meet SP, $S'P$ in E and E' (fig. Art. 66), and the normal at P in F; then SPY, PEF, and $S'PY'$ are similar triangles;

$$\therefore\ SP : SY :: PE : PF,$$

and $$S'P : S'Y' :: PE : PF;$$

$$\therefore\ SP\,.\,S'P : SY\,.\,S'Y' :: PE^2 : PF^2$$
$$:: AC^2 : PF^2$$
$$:: CD^2 : BC^2,\ \text{Art. 80};$$
$$\therefore\ SP\,.\,S'P = CD^2.$$

82. **Prop. XXVI.** *If the tangent at P meet a pair of conjugate diameters in T and t, and CD be conjugate to CP,*

$$PT\,.\,Pt = CD^2.$$

Taking the figure of Art. 77,

$$PT : PN :: CD : DM;$$

and, if TP produced meet CB in t,

$$Pt : CN :: CD : CM;$$

$$\therefore\ PT\,.\,Pt : PN\,.\,CN :: CD^2 : DM\,.\,CM.$$

But $$PN\,.\,CN = DM\,.\,CM,\ \text{Art. 77},$$

$$\therefore\ PT\,.\,Pt = CD^2.$$

Cor. Let TQV be the tangent at the other end of the chord PNQ, meeting CB' produced in U; and let CE be the semidiameter parallel to TQ.

Then $$TP : TQ :: Pt : QU,$$

$$\therefore\ TP^2 : TQ^2 :: PT\,.\,Pt : QT\,.\,QU$$
$$:: CD^2 : CE^2,$$

that is, *the two tangents drawn from any point are in the ratio of the parallel diameters.*

83. In a similar manner it can be shewn that, if the tangent at P meet the tangents at the ends of a diameter ACA' in T and T',

$$PT \,.\, PT' = CD^2,$$

CD being conjugate to CP,

and
$$AT \,.\, AT' = CB^2,$$

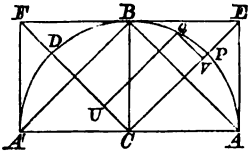
CB being conjugate to ACA'.

These properties can be demonstrated by the help of Art. 77, and of the corollary to Art. 82.

84. *Equi-conjugate diameters.*

Prop. XXVII. *The diagonals of the rectangle formed by the principal axes are equal and conjugate diameters.*

For, joining AB, $A'B$, these lines are parallel to the diagonals CF, CE; and, AB, $A'B$ being supplemental chords, it follows that CD, CP are conjugate to each other. Moreover, they are equally inclined to the axes, and are therefore of equal length.



Cor. 1. If QV, QU be drawn parallel to the equi-conjugate diameters, meeting them in V and U,

$$QV^2 : CP^2 - CV^2 :: CD^2 : CP^2;$$

$$\therefore QV^2 = CP^2 - CV^2 = PV \,.\, VP',$$

if P' be the other end of the diameter PCP'.

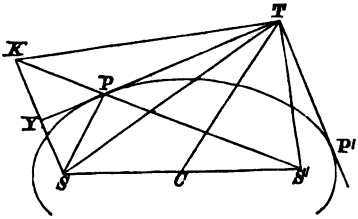
Hence
$$QV^2 + QU^2 = CP^2.$$

Cor. 2.
$$CP^2 + CD^2 = AC^2 + BC^2, \text{ (Art. 78)};$$

$$\therefore 2CP^2 = AC^2 + BC^2.$$

85. Prop. XXVIII. ***Pairs of tangents at right angles to each other intersect on a fixed circle.***

The two tangents being TP, TP', let $S'P$ produced meet SY the perpendicular on TP in K.



Then the angle $PTK = STP = S'TP'$;

$$\therefore\ S'TK \text{ is a right angle.}$$

Hence
$$\begin{aligned} 4AC^2 &= S'K^2 = S'T^2 + TK^2 \\ &= S'T^2 + ST^2 \\ &= 2CT^2 + 2CS^2, \text{ Euclid, II. 12;} \end{aligned}$$

$$\therefore\ CT^2 = AC^2 + BC^2,$$

and T lies on a fixed circle, centre C.

This circle is called the *Director Circle* of the Ellipse, and it will be seen that when the ellipse, by the elongation of SC from S is transformed into a parabola, the director circle merges into the directrix of the parabola.

86. Prop. XXIX. ***The rectangles contained by the segments of any two chords which intersect each other are in the ratio of the squares of the parallel diameters.***

Through any point O in a chord OQQ' draw the diameter ORR', and let CD be parallel to QQ', and CP conjugate to CD, bisecting QQ' in V.

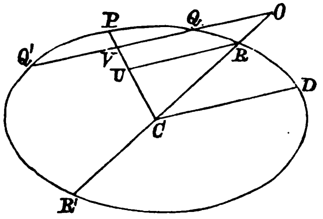
Draw RU parallel to CD.

Then $CD^2 - RU^2 : CU^2 :: CD^2 : CP^2$, Art. 76,

$$:: CD^2 - QV^2 : CV^2.$$

But $RU^2 : CU^2 :: OV^2 : CV^2$;

$$\therefore CD^2 : CU^2 :: CD^2 + OV^2 - QV^2 : CV^2,$$



or $CD^2 : CD^2 + OV^2 - QV^2 :: CU^2 : CV^2$

$$:: CR^2 : CO^2;$$

$$\therefore CD^2 : OV^2 - QV^2 :: CR^2 : CO^2 - CR^2,$$

or $CD^2 : OQ.OQ' :: CR^2 : OR.OR'$.

Similarly if Oqq' be any other chord through O, and Cd the parallel semi-diameter,

$$Cd^2 : Oq.Oq' :: CR^2 : OR.OR';$$

$$\therefore OQ.OQ' : Oq.Oq' :: CD^2 : Cd^2.$$

This may otherwise be expressed thus,

The ratio of the rectangles of the segments depends only on the directions in which they are drawn.

The proof is the same if the point O be within the ellipse.

87. Prop. XXX. *If a circle intersect an ellipse in four points the several pairs of the chords of intersection are equally inclined to the axes.*

For if QQ', qq' be a pair of the chords of intersection, and if these meet in O, or be produced to meet in O, the rectangles $OQ . OQ'$, $Oq . Oq'$ are proportional to the squares on the parallel diameters.

But these rectangles are equal since QQ', qq' are chords of a circle.

Therefore the parallel diameters are equal, and, since equal diameters are equally inclined to the axes, it follows that the chords QQ', qq' are equally inclined to the axes.

88. Conversely, if two chords, not parallel, be equally inclined to the axes a circle can be drawn through their extremities.

For, as in Art. 87, if OQQ' be a chord,

$$OQ . OQ' : OR . RR' :: CD^2 : CR^2,$$

and if Oqq' be another chord, and Cd the semi-diameter parallel to it,

$$Oq . Oq' : OR . OR' :: Cd^2 : CR^2;$$

but, if CD and Cd be equally inclined to the axes, they are equal, and

$$\therefore OQ . OQ' = Oq . Oq',$$

and a circle can be drawn through the points Q, Q', q, and q'.

EXAMPLES.

1. If the tangent at B meet the latus rectum produced in D, CDX is a right angle.

2. If PCp be a diameter, and the focal distance pS produced meet the tangent at P in T, $SP = ST$.

3. If the normal at P meet the axis minor in G' and $G'N$ be the perpendicular from G' on SP, then $PN = AC$.

4. TQ, TQ' are two tangents at right angles, and CT meets QQ' in V; prove that $VT = QV$, and, by help of this equality, shew that the locus of T is a circle.

5. The tangent at P bisects any straight line perpendicular to AA' and terminated by AP, $A'P$, produced if necessary.

6. Draw a tangent to an ellipse parallel to a given line.

7. SR being the semi-latus rectum, if RA meet the directrix in E, and SE meet the tangent at A in T,

$$AT = AS.$$

8. Prove that $SY : SP :: SR : PG$.

Find where the angle SPS' is greatest.

9. If two points E and E' be taken in the normal PG such that $PE = PE' = CD$, the loci of E and E' are circles.

Shew that the sum of two conjugate diameters is greater than the sum of the axes, and that their difference is less than the difference of the axes.

10. If from the focus S' a line be drawn parallel to SP, it will meet the perpendicular SY in the circumference of a circle.

11. If the normal at P meet the axis major in G, prove that PG is an harmonic mean between the perpendiculars from the foci on the tangent at P.

12. If tangents TP, TQ be drawn at the extremities P, Q of any focal chord of an ellipse, prove that the angle PTQ is half the supplement of the angle which PQ subtends at the other focus.

13. If Y, Z be the feet of the perpendiculars from the foc on the tangent at P; prove that the circle circumscribed about the triangle YNZ will pass through C.

14. If AQ be drawn from one of the vertices perpendicular to the tangent at any point P, prove that the locus of the point of intersection of PS and QA produced will be a circle.

15. If the normal at P meet the axis major in G and the axis minor in K, prove that a circle can be drawn through the foci and through the points P, K, and that $GK : SK :: SA : AX$; shew also that, if the tangent at P meet the axis minor in t,

$$St : tK :: BC : CD,$$

CD being conjugate to CP.

16. The straight lines joining each focus to the foot of the perpendicular from the other focus on the tangent at any point meet on the normal at the point and bisect it.

17. If two circles touch each other internally, the locus of the centres of circles touching both is an ellipse whose foci are the centres of the given circles.

18. The subnormal at any point P is a third proportional to the intercept of the tangent at P on the major axis and half the minor axis.

19. If the normal at a point P of an ellipse meet the axis in G, and the tangent at P meet the axis in T, prove that

$$TQ : TP :: BC : PG,$$

Q being the point where the ordinate at P meets the auxiliary circle.

20. If the tangent at any point P of an ellipse meet the tangent at the extremities of the axis AA' in F and F', prove that the rectangle AF, $A'F'$ is equal to the square on the semi-axis minor.

21. From any point T outside an ellipse tangents TP, TQ are drawn to the ellipse; prove that a circle can be described with T as centre so as to touch SP, HP, SQ, and HQ, or these lines produced, S and H being the foci.

22. If two equal and similar ellipses have the same centre, shew that their points of intersection are at the extremities of diameters at right angles to one another.

23. The external angle between any two tangents to an ellipse is equal to the semi-sum of the angles which the chord joining the points of contact subtends at the foci.

24. The tangent at any point P of an ellipse meets the axes in T and t; if S be a focus the angles PSt, STP are equal.

25. P is a point in an ellipse, PM, PN perpendicular to the axes meet respectively when produced the circles described on these axes as diameters in the points QQ_1, shew that QQ_1 passes through the centre.

26. In an ellipse if SY is a perpendicular from the focus S on the tangent at P and CD a diameter conjugate to CP,

$$SY \,.\, CD = SP \,.\, BC.$$

27. A conic is drawn touching an ellipse at the extremities A, B of the axes, and passing through the centre C of the ellipse; prove that the tangent at C is parallel to AB.

28. The tangent to an ellipse at any point P is cut by any two conjugate diameters in T, t, and the points T, t, are joined with the foci S, H respectively; prove that the triangles SPT, HPt are similar to each other.

29. If the diameter conjugate to CP meet SP, and HP (or these produced) in E and E', prove that SE is equal to HE', and that the circles which circumscribe the triangles SCE', HCE, are equal to one another.

30. PG is a normal to an ellipse, terminating in the major axis; the circle, of which PG is a diameter, cuts SP, HP, in K, L, respectively: prove that KL is bisected by PG, and is perpendicular to it.

31. S and H are the foci of an ellipse, and P is a point on the curve; prove that the locus of the centre of the circle inscribed in the triangle SPH is an ellipse.

32. Tangents to an ellipse are drawn from any point in a circle through the foci, prove that the lines bisecting the angle between the tangents all pass through a fixed point.

33. If a quadrilateral circumscribe an ellipse, the angles subtended by opposite sides at one of the foci are together equal to two right angles.

34. If the normal at P in an ellipse meet the axis minor in G, and if the tangent at P meet the tangent at the vertex A in V, shew that $SG : SC :: PV : VA$.

35. P, Q are points in two confocal ellipses, at which the line joining the common foci subtends equal angles; prove that the tangents at P, Q are inclined at an angle which is equal to the angle subtended by PQ at either focus.

36. The transverse axis is the greatest and the conjugate axis the least of all the diameters.

37. If P, P' be any two points of an ellipse, and S, S' the foci, prove that a circle can be drawn so as to touch the four straight lines, SP, SP', $S'P$, $S'P'$.

38. If SY, $S'Y'$ be the perpendiculars from the foci on the tangent at a point P of an ellipse, and PN the ordinate, prove that

$$PY : PY' :: NY : NY'.$$

39. If a circle, passing through Y and Z, touch the major axis in Q, and that diameter of the circle, which passes through Q, meet the tangent in P, then $PQ = BC$.

40. Two common chords of a given ellipse and a circle pass through a given point; shew that the locus of the centres of all such circles is a straight line through the given point.

41. Two conjugate diameters of an ellipse are cut by the tangent at any point P in M, N; prove that the area of the triangle CPM varies inversely as that of the triangle CPN.

42. If P be any point of an ellipse, centre C and AV be drawn parallel to PC to meet the conjugate CD in V, prove that the areas of the triangles CAV, CPN are equal, PM being the ordinate.

43. CP and CD are conjugate semi-diameters of an ellipse; PQ is a chord parallel to one of the axes: shew that DQ is parallel to one of the straight lines which join the ends of the axes.

44. Prove that the distance between the two points on the circumference of an ellipse, at which a given chord, not passing through the centre, subtends the greatest and least angles, is equal to the diameter which bisects that chord.

45. A tangent to an ellipse at a point P intersects a fixed tangent in T; if S is the focus and a line be drawn through S perpendicular to ST, meeting the tangent at P in Q, shew that the locus of Q is a straight line touching the ellipse.

46. Shew that, if the distance between the foci of the ellipse be greater than the length of its axis minor, there will be four positions of the tangent, for which the area of the triangle, included between it and the straight lines drawn from the centre of the curve to the feet of the perpendiculars from the foci on the tangent, will be the greatest possible.

47. Two ellipses whose axes are equal, each to each, are placed in the same plane with their centres coincident, and axes inclined to each other. Draw their common tangents.

48. If Pp be drawn parallel to the transverse axis meeting the ellipse in P, p, and the circle whose diameter is the conjugate axis in R, r, then shall

$$Pp : Rr :: QQ' : PP'.$$

49. The tangent at any point P of a circle meets the tangent at a fixed point A in T, and T is joined with B the extremity of the diameter passing through A; the locus of the point of intersection of AP, BT is an ellipse.

50. If PG, the normal at P, cut the major axis in G, and if DR, PN be the ordinates of D and P, CD being conjugate to CP, prove that the triangles PGN, DRC are similar; and thence deduce that PG bears a constant ratio to CD.

51. The ordinate NP at a point P of an ellipse meets, when produced, the circle on the major axis in Q. If S be a focus of the ellipse, prove geometrically, or otherwise, that $SQ : SP ::$ the axis major : the chord of the circle through Q and S, and that the diameter of the ellipse parallel to SP is equal to the same chord.

52. If the perpendicular from the centre C of a conic on the tangent at P meet the focal distance SP produced in R, the locus of R is a circle, the diameter of which is equal to the axis major.

53. A perfectly elastic billiard ball lies on an elliptical billiard table, and is projected in any direction along the table shew that all the lines in which it moves after each successive impact touch an ellipse or an hyperbola confocal with the billiard table.

54. Shew that a circle can be drawn through the foci and the intersections of any tangent with the tangent at the vertices.

55. Any diameter is a mean proportional between the transverse axis and the focal chord parallel to the diameter.

56. If CP, CD be conjugate semi-diameters, and a rectangle be described so as to have PD for a diagonal and its sides parallel to the arcs, the other angular points will be situated on two fixed straight lines passing through the centre C.

57. If the tangent at P meet the minor axis in T, prove that the areas of the triangles SPS', STS' are in the ratio of the squares on CD and ST.

58. Find the locus of the centre of the circle touching the transverse axis, SP, and $S'P$ produced.

59. If a parallelogram circumscribe an ellipse, it is not less than the rectangle contained by the axes.

60. If T be the angle between two tangents at the ends of a focal chord, and S the angle subtended by the chord at the other focus, $2T+S$ is equal to two right angles.

61. If CD be conjugate to CP,

$$SY : SP :: BC : CD.$$

62. If PSP', QSQ' be focal chords,

$$PS \,.\, SP' : QS \,.\, SQ' :: PP' : QQ'.$$

63. If the centre, a tangent, and the transverse axis of an ellipse be given, prove that the directrices pass each through a fixed point.

64. The straight line joining the feet of perpendiculars from the focus on two tangents is at right angles to the line joining the intersection of the tangents with the other focus.

65. A circle passes through a focus of an ellipse, has its centre on the major axis of the ellipse, and touches the ellipse: shew that the straight line from the focus to the point of contact is equal to the latus rectum.

66. Prove that the perimeter of the quadrilateral formed by the tangent, the perpendiculars, and the transverse axis, will be the greatest possible when the focal distances of the point of contact are at right angles to each other.

67. If CQ be conjugate to the normal at P in an ellipse, then is CP conjugate to the normal at Q.

68. From any point on one of the equi-conjugate diameters of an ellipse two tangents are drawn; prove that the circle passing through the point and the two points of contact will also pass through the centre.

69. If PN be the ordinate of any point P of an ellipse, centre C, and if with centre C and radius equal to PN a circle be described intersecting PN in Q, prove that the locus of Q is an ellipse.

70. If CP be any diameter of an ellipse, and AQO be drawn parallel to CP, meeting the curve in Q and the minor axis in O, $2CP^2 = AO \, . \, AQ$.

71. PS is the focal distance of a point in an ellipse; CR is a radius of the auxiliary circle parallel to PS, and drawn in the direction from P to S; SQ is a perpendicular on CR: shew that the rectangle contained by SP and QR is equal to the square on half the minor axis.

72. If a focus of an ellipse be joined with the point where the tangent at the nearer vertex intersects any other tangent, and perpendiculars be let fall from the other focus on the joining line and on the last-mentioned tangent, prove that the distance between the feet of these perpendiculars is equal to the distance from either focus to the remoter vertex.

73. A parallelogram is described about an ellipse; if two of its angular points lie on the directrices, the other two will lie on the auxiliary circle.

74. From a point in the auxiliary circle straight lines are drawn touching the ellipse in P and P'; prove that SP is parallel to $S'P'$.

75. If the tangent and normal at a point P of an ellipse meet the axis major in T and G respectively, prove that

$$CG \, . \, CT = CS^2.$$

76. Find the locus of the points of contact of tangents to a series of confocal ellipses from a fixed point in the axis major.

77. A series of confocal ellipses intersect a given straight line; prove that the locus of the points of intersection of the pairs of tangents drawn at the extremities of the chords of intersection is a straight line at right angles to the given straight line.

78. Given the focus and the length of the major axis; describe an ellipse touching a given straight line and passing through a given point.

79. Given the focus and the length of the major axis; describe an ellipse touching two given straight lines.

80. Find the positions of the foci and directrices of an ellipse which touches at two given points P, Q, two given straight lines PO, QO, and has one focus on the line PQ, the angle POQ being less than a right angle.

81. Through any point P of an ellipse are drawn straight lines APQ, $A'PR$, meeting the auxiliary circle in Q, R, and ordinates Qq, Rr are drawn to the transverse axis; prove that, L being an extremity of the latus rectum,

$$Aq \,.\, A'r : Ar \,.\, A'q :: AC^2 : SL^2.$$

82. If a tangent at a point P meet the major axis in T, and the perpendiculars from the focus and centre in Y and Z, then

$$TY^2 : PY^2 :: TZ : PZ.$$

83. An ellipse slides between two lines at right angles to each other; find the locus of its centre.

84. TP, TQ are two tangents to an ellipse, and CP', CQ' are the radii from the centre respectively parallel to these tangents, prove that $P'Q'$ is parallel to PQ.

85. The tangent at P meets the minor axis in t; prove that

$$St \,.\, PN = BC \,.\, CD.$$

86. If the circle, centre t, and radius tS, meet the ellipse in Q, and QM be the ordinate, prove that

$$QM : PN :: BC : BC + CD.$$

87. OQ, OQ' are tangents to an ellipse from an external point O, and OR is a diagonal of the parallelogram of which OQ, OQ' are adjacent sides; prove that if R be on the ellipse, O will lie on a similar and similarly situated concentric ellipse.

88. PQ is the chord of an ellipse normal at P, LCL' the diameter bisecting it, shew that PQ bisects the angle LPL' and $LP+PL'$ is constant.

89. ABC is an isosceles triangle of which the side AB is equal to the side AC. BD, BE drawn on opposite sides of BC and equally inclined to it meet AC in D and E. If an ellipse is described round BDE having its axis minor parallel to BC, then AB will be a tangent to the ellipse.

90. If A be the extremity of the major axis of an ellipse and P any point on the curve, the bisectors of the angles PSA, $PS'A$ meet on the tangent at P.

91. If two ellipses intersect in four points, the diameters parallel to a pair of the chords of intersection are in the same ratio to each other.

92. From any point P of an ellipse a straight line PQ is drawn perpendicular to the focal distance SP, and meeting in Q the diameter conjugate to that through P; shew that PQ varies inversely as the ordinate of P.

93. The tangent at a point P of an ellipse meets the tangents at the vertices in V, V'; on VV' as diameter a circle is described, which intersects the ellipse in Q, Q'; shew that the ordinate to Q : the ordinate to P :: BC : $BC+CD$, where CD is conjugate to CP.

94. PCP' is any diameter of an ellipse. The tangents at any two points D and E intersect in F. PE, $P'D$ intersect in G. Shew that FG is parallel to the diameter conjugate to PCP'.

95. If a circle be drawn through the foci of two confocal ellipses, cutting the ellipses in P and Q, the tangents to the ellipses at P and Q will intersect on the circumference of the same circle.

96. If any two points P, Q be given in an ellipse, prove that a third point R may be found so that the angle PRQ is a maximum by the following construction. Draw a tangent parallel to

PQ, touching the ellipse in *K*, and draw *KR* perpendicular to the major axis, cutting the curve again in *R*.

97. Through the middle point of a focal chord of an ellipse a straight line is drawn at right angles to it to meet the axis in *R*; prove that *SR* bears to *SC* the duplicate ratio of the chord to the diameter parallel to it, *S* being the focus and *C* the centre.

98. The tangent at a point *P* of an ellipse meets the auxiliary circle in Q' to which corresponds *Q* on the ellipse; prove that the tangent at *Q* cuts the auxiliary circle in the point corresponding to *P*.

99. If a chord be drawn to a series of concentric, similar, and similarly situated ellipses, and meet one in *P* and *Q*, and if on *PQ* as diameter a circle be described meeting that ellipse again in *RS*, shew that *RS* is constant in position.

100. An ellipse touches the sides of a triangle; prove that if one of its foci move along the arc of a circle passing through two of the angular points of the triangle, the other will move along the arc of a circle through the same two angular points.

CHAPTER IV.

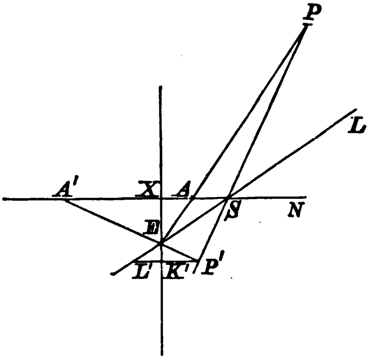
THE HYPERBOLA.

DEFINITION.

An hyperbola is the curve traced by a point which moves in such a manner, that its distance from a given point is in a constant ratio of greater inequality to its distance from a given straight line.

Tracing the Curve.

89. Let S be the focus, EX the directrix, and A the vertex.



Then, as in Art. 2, any number of points on the curve may be obtained by taking successive positions of E on the directrix.

In SX produced, find a point A' such that

$$SA' : A'X :: SA : AX,$$

then A' is the other vertex as in the ellipse, and, the eccentricity being greater than unity, the points A and A' are evidently on opposite sides of the directrix.

Find the point P corresponding to E, and let $A'E$, PS produced meet in P', then, if $P'K'$ perpendicular to the directrix meet SE produced in L',

$$\begin{aligned} P'L' : P'K' &:: SA' : A'X \\ &:: SA \;\, : AX, \end{aligned}$$

and the angle

$$P'L'S = L'SX = L'SP';$$

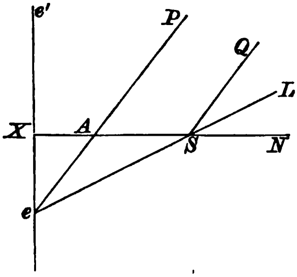
$$\therefore\ SP' = P'L'.$$

Hence P' is a point in the curve, and PSP' is a focal chord.

Following out the construction, we observe that, since SA is greater than AX, there are two points on the directrix, e and e', such that Ae and Ae' are each equal to AS.

If E coincide with e, the angle

$$QSL = LSN = ASe = AeS.$$



Hence SQ, AP are parallel, and the corresponding point of the curve is at an infinite distance; and similarly the curve tends to infinity in the direction Ae'.

Further, the angle ASE is less or greater than AES, according as the point E is, or is not, between e and e'.

Hence, when E is below e, the curve lies above the axis, to the right of the directrix; when between e and X, below the axis, to the left; when between X and e', above the axis to the left; and when above e', below the axis, to the right. Hence a general idea can be obtained of the form of the curve, tending to infinity in four directions, as in the figure of Art. 98.

DEFINITIONS.

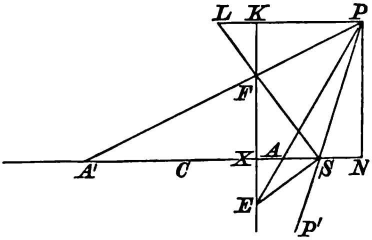
The line AA' is called the transverse axis of the hyperbola.

The middle point, C, of AA' is the centre.

Any straight line, drawn through C, and terminated by the curve is called a diameter.

90. PROP. I. *If P be any point of an hyperbola, and AA' its transverse axis, and if $A'P$, and PA produced, (or PA and PA' produced) meet the directrix in E and F, EF subtends a right angle at the focus.*

Let PKL, perpendicular to the directrix, meet SF produced in L.



Then . $$PL : SA' :: PF : A'F$$
$$:: PK : A'X,$$
or $$PL : PK :: SA' : A'X;$$
$$\therefore SP = PL,$$

and the angle

$$LSP = PLS = LSX.$$

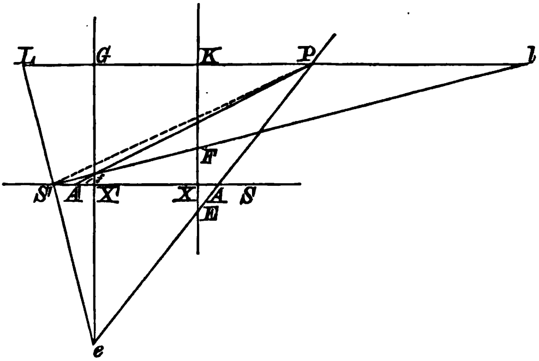
But, if PS be produced to P', ES bisects the angle ASP';

$$\therefore\ ESF \text{ is a right angle.}$$

91. We shall now prove the existence of another focus and directrix corresponding to the vertex A'.

In AA' take a point X' such that $A'X' = AX$, and through X' draw a straight line perpendicular to AA'. Also in SA' produced, take a point S' such that $A'S' = AS$.

Let $A'P$ and PA produced meet the perpendicular through X' in f and e, and join $S'e$, $S'f$.



Then $eX' : EX :: AX' : AX$,

$$:: A'X : A'X',$$

$$:: FX : fX';$$

$$\therefore\ eX' \,.\, fX' = EX \,.\, FX = SX^2 = S'X'^2.$$

Hence $eS'f$ is a right angle.

Through P draw PKG parallel to the axis meeting EX, eX' in K, G, and eS', $S'f$ produced in L and l;

then $$PL : PG :: S'A : AX',$$

and $$Pl : PG :: S'A' : A'X';$$

$$\therefore PL = Pl;$$

and, $LS'l$ being a right angle,

$$S'P = PL = Pl;$$

$$\therefore S'P : PG :: S'A' : A'X';$$

and the curve can be described by means of the focus S' and directrix eX'.

Hence it follows that the curve is symmetrical with regard to the point C, and that it lies wholly without the tangents at the vertices A and A'.

92. Prop. II. *If PN be the ordinate of a point P, and ACA' the transverse axis, PN^2 is to $AN.NA'$ in a constant ratio.*

Join AP, $A'P$, meeting the directrix in E and F, fig. Art. 90.

Then $$PN : AN :: EX : AX,$$

and $$PN : A'N :: FX : A'X;$$

$$\therefore PN^2 : AN.NA' :: EX.FX : AX.A'X$$
$$:: SX^2 : AX.A'X.$$

Since ESF is a right angle; that is, PN^2 is to $AN.NA'$ in a constant ratio.

Through C, the middle point of AA', draw CB at right angles to the axis, and such that

$$BC^2 : AC^2 :: SX^2 : AX.A'X;$$

then $$PN^2 : AN.NA' :: BC^2 : AC^2,$$

or $$PN^2 : CN^2 - AC^2 :: BC^2 : AC^2.$$

Cor. If PM be the perpendicular from P on BC,

$$PM = CN, \text{ and } PN = CM;$$

$$\therefore CM^2 : PM^2 - AC^2 :: BC^2 : AC^2,$$

or $$CM^2 : BC^2 :: PM^2 - AC^2 : AC^2;$$

$$\therefore CM^2 + BC^2 : BC^2 :: PM^2 : AC^2,$$

or $$PM^2 : CM^2 + BC^2 :: AC^2 : BC^2.$$

Conversely, if a point P curve in such a manner that PN^2 is to $AN \,.\, NA'$ in a constant ratio, PN being the distance of P from the line joining two fixed points A and A', and N not being between A and A', the locus of P is an hyperbola of which AA' is the transverse axis.

93. If we describe the circle on AA' as diameter, which we may term for convenience, *the auxiliary circle*, the rectangle $AN \,.\, NA'$ is equal to the square on the tangent to the circle from N.

Hence the preceding theorem may be thus expressed:

The ordinate of an hyperbola is to the tangent from its foot to the auxiliary circle in the ratio of the conjugate to the transverse axis.

Def. *If CB' be taken equal to CB, on the other side of the axis, the line BCB' is called the conjugate axis.*

The two lines AA', BB' are the principal axes of the curve.

When these lines are equal, the hyperbola is said to be equilateral, or rectangular.

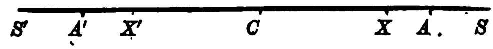
The lines AA', BB' are sometimes called major and minor axes, but, as AA' is not necessarily greater than BB', these terms cannot with propriety be generally employed.

94. Prop. III. *If ACA' be the transverse axis, C the centre, S one of the foci, and X the foot of the directrix,*

$$CS : CA :: CA : CX :: SA : AX,$$

and $$CS : CX :: CS^2 : CA^2.$$

Interchanging the positions of S and X for a new



figure, the proof of these relations is identical with the proof given for the ellipse in Art. 57.

95. Prop. IV. *If S be a focus, and B an extremity of the conjugate axis.*

$$BC^2 = AS \, . \, SA', \text{ and } SC^2 = AC^2 + BC^2.$$

Referring to Art. (94), $SX = SA + AX$;

$$\therefore SX : AX :: SA + AX : AX,$$

$$:: SC + AC : AC;$$

and similarly

$$SX : A'X :: SC - AC : AC;$$

$$\therefore SX^2 : AX \, . \, A'X :: SC^2 - AC^2 : AC^2.$$

But $BC^2 : AC^2 :: SX^2 : AX \, . \, A'X$;

$$\therefore BC^2 = SC^2 - AC^2 = AS \, . \, SA'.$$

Hence $SC^2 = AC^2 + BC^2 = AB^2$;

i. e. SC is equal to the line joining the ends of the axes.

96. Prop. V. *The difference of the focal distances of any point is equal to the transverse axis.*

For, if PKK', perpendicular to the directrices, meet them in K and K',

$$S'P : PK' :: SA : AX,$$

and $SP : PK :: SA : AX$;

$$\therefore S'P - SP : KK' :: SA : AX,$$

$$:: AA' : XX', \text{ Art. 93.}$$

$$\therefore S'P - SP = AA'.$$

Cor. 1. $SP : NX :: AC : CX$;

$$\therefore SP : AC :: NX : CX;$$

$$\therefore SP + AC : AC :: CN : CX,$$

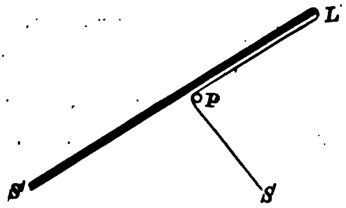
or $SP + AC : CN :: SA : AX.$

Hence also $S'P - AC : CN :: SA : AX.$

COR. 2. *Hence also it can be easily shewn, that the difference of the distances of any point from the foci of an hyperbola, is greater or less than the transverse axis, according as the point is within or without the concave side of the curve.*

97. *Mechanical Construction of the Hyperbola.*

Let a straight rod $S'L$ be moveable in the plane of the paper about the point S'. Take a piece of string, the length of which is less than that of the rod, and fasten one end to a fixed point S, and the other end to L, then pressing a pencil against the string so as to keep it stretched, and a part of it PL in contact with the rod, the pencil will trace out on the paper an hyperbola, having its foci at S and S', and its transverse axis equal to the difference between the length of the rod and that of the string.



This construction gives the right-hand branch of the curve; to trace the other branch, take the string longer than the rod, and such that it exceeds the length of the rod by the transverse axis.

We may remark that by taking a longer rod $MS'L$, and taking the string longer than $SS'+S'L$, so that the point P will be always on the end $S'M$ of the rod, we shall obtain an ellipse of which S and S' are the foci. Moreover, remembering that a parabola is the limiting form of an ellipse when one of the foci is removed to an infinite distance, the mechanical construction, given for the parabola will be seen to be a particular case of the above.

The Asymptote.

98. We have shewn in Art. (89) that if two points, e and e', be taken on the directrix such that

$$Ae = Ae' = AS,$$

the lines eA, $e'A$ meet the curve at an infinite distance.

These lines are parallel to the diagonals of the rectangle formed by the axes, for

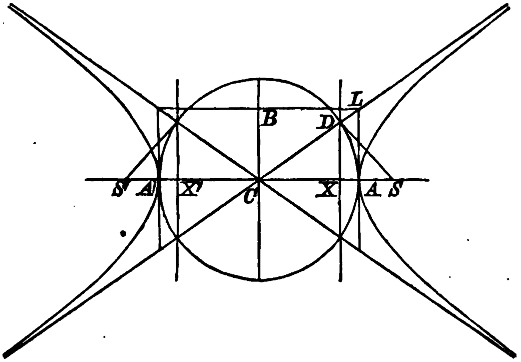
$$\begin{aligned} Ae : AX &:: AS : AX, \\ &:: SC : AC, \text{ Art. (93)}, \\ &:: AB : AC. \end{aligned}$$

DEFINITION. *The diagonals of the rectangle formed by the principal axis are called the asymptotes.*

We observe that the axes bisect the angles between the asymptotes, and that if a double ordinate, PNP', when produced, meet the asymptotes in Q and Q',

$$PQ = P'Q'.$$

The figure appended will give the general form of the curve and its connection with the asymptotes and the auxiliary circle.



99. **Prop. VI.** *The asymptotes intersect the directrices in the same points as the auxiliary circle, and the lines joining the corresponding foci with the points of intersection are tangents to the circle.*

Let the asymptote CL meet the directrix in D, and join SD; fig. Art. (98).

$$\text{Then} \quad CD : CX :: CL : CA$$
$$:: SC : CA, \text{ Art. 94},$$
$$:: CA : CX;$$

$$\therefore CD = CA, \text{ and } D \text{ is on the auxiliary circle.}$$

Also

$$CS \,.\, CX = CA^2 = CD^2;$$

$\therefore$ CDS is a right angle, and SD is the tangent at D.

Cor. $CD^2 + SD^2 = CS^2 = AC^2 + BC^2$, Art. 95;

$$\therefore SD = BC.$$

100. An asymptote may also be characterized as the ultimate position of a tangent when the point of contact is removed to an infinite distance.

It appears from Art. (7) that in order to find the point of contact of a tangent drawn from a point T in the directrix, we must join T with the focus S, and draw through S a straight line at right angles to ST; this line will meet the curve in the point of contact.

In the figure of Art. (98) we know that the straight line through S, parallel to eA or CL, meets the curve in a point at an infinite distance, and also that this straight line is at right angles to SD, since SD is at right angles to CD. Hence the tangent from D, that is the line from D to the point at an infinite distance, is perpendicular to DS and therefore coincident with CD.

The asymptotes therefore touch the curve at an infinite distance.

101. **Def.** *If an hyperbola be described, having for its transverse and conjugate axes, respectively, the conjugate and transverse axes of a given hyperbola, it is called the conjugate hyperbola.*

It is evident from the preceding article that the conjugate hyperbola has the same asymptotes as the original hyperbola, and that the distances of its foci from the centre are also the same.

The relations of Art. (92) and its Corollary are also true, *mutatis mutandis*, of the conjugate hyperbola; thus, in Art. (92), if P be a point in the conjugate hyperbola,

$$PM^2 : CM^2 - BC^2 :: AC^2 : BC^2,$$

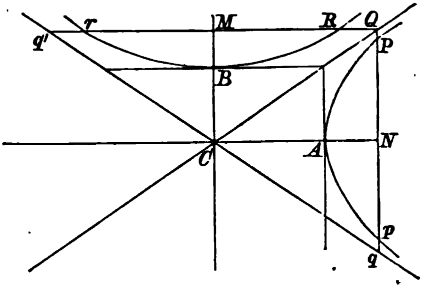
and
$$CM^2 : PM^2 + AC^2 :: BC^2 : AC^2.$$

Def. *A straight line drawn through the centre and terminated by the conjugate hyperbola is also called a diameter of the original hyperbola.*

102. Prop. VII. *If from any point Q in one of the asymptotes, two straight lines QPN, QRM be drawn at right angles respectively to the transverse and conjugate axes, and meeting the hyperbola in P, p, and the conjugate hyperbola in R, r,*

$$QP \,.\, Qp = BC^2,$$

and
$$QR \,.\, Qr = AC^2.$$



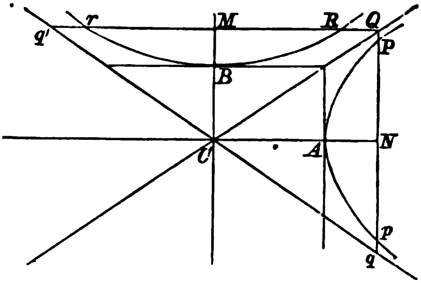
For $$QN^2 : BC^2 :: CN^2 : AC^2;$$

$$\therefore QN^2 - BC^2 : BC^2 :: CN^2 - AC^2 : AC^2$$
$$:: PN^2 : BC^2;$$

$$\therefore \; QN^2 - BC^2 = PN^2,$$

or $$QN^2 - PN^2 = BC^2;$$

i. e. $$QP \, . \, Qp = BC^2.$$



Similarly, $QM^2 : AC^2 :: CM^2 : BC^2$;

$$\therefore \; QM^2 - AC^2 : AC^2 :: CM^2 - BC^2 : BC^2,$$

$$:: RM^2 : AC^2;$$

$$\therefore \; QM^2 - RM^2 = AC^2,$$

or $$QR \, . \, Qr = AC^2.$$

These relations may also be given in the form,

$$QP \, . \, Pq = BC^2, \qquad QR \, . \, Rq' = AC^2.$$

Cor. If the point Q be taken at a greater distance from C, the length QN and therefore Qp will be increased and may be increased indefinitely.

But the rectangle $QP \, . \, Qp$ is of finite magnitude; hence QP will be indefinitely diminished, and the curve, therefore, as it recedes from the centre, tends more and more nearly to coincidence with the asymptote.

A further illustration is thus given of the remarks in Art. (100).

103. If in the preceding figure the line Qq be produced to meet the conjugate hyperbola in E and e, it can be shewn, in the same manner as in Art. (102),

$$QE \, . \, Qe = BC^2;$$

and this equality is still true when the line Qq lies between C and A, in which case Qq does not meet the hyperbola.

Properties of the Tangent and Normal.

104. Prop. VIII. *The tangent at any point bisects the angle between the focal distances of that point, and the normal is equally inclined to the focal distances.*

Let the normal at P meet the axis in G.

Then, Art. (15),

$$SG : SP :: SA : AX,$$

and $$S'G : S'P :: SA : AX;$$

$$\therefore SG : S'G :: SP : S'P;$$

and therefore the angle between SP and $S'P$ produced is bisected by PG.

Hence PT the tangent which is perpendicular to PG bisects the angle SPS'.

Cor. 1. If PT and GP produced meet, respectively, the conjugate axis in t and g, it can be shewn, in exactly the same manner as in the corresponding case of the ellipse, Art. 64, that the circle which circumscribes SPS' also passes through t and g.

Cor. 2. If an ellipse be described having S and S' for its foci, and if this ellipse meet the hyperbola in P, the normal at P to the ellipse bisects the angle SPS', and therefore coincides with the tangent to the hyperbola.

Hence, if an ellipse and an hyperbola be confocal, that is, have the same foci, they intersect at right angles.

105. Prop. IX. *Every diameter is bisected at the centre, and the tangents at the ends of a diameter are parallel.*

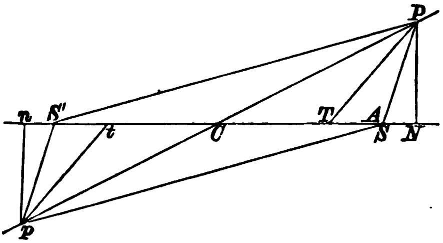
Let PCp be a diameter, and PN, pn the ordinates.

Then $$CN^2 : Cn^2 :: PN^2 : pn^2,$$

$$:: CN^2 - AC^2 : Cn^2 - AC^2;$$

hence $CN = Cn$,

and $\therefore\ CP = Cp$.



Again, if PT, pt be the tangents,

The triangles PCS, pCS' are equal in all respects, and therefore $SPS'p$ is a parallelogram.

Hence the angles SPS', SpS' are equal, and therefore $SPT = S'pt$.

But $SPC = S'pC$,

$\therefore$ the difference TPC = the difference tpC, and PT is parallel to pt.

It can be shewn in exactly the same manner, that, if the diameter be terminated by the conjugate hyperbola, it is bisected in C, and the tangents at its extremities are parallel.

Cor. The distances SP, Sp are equally inclined to the tangents at P and p.

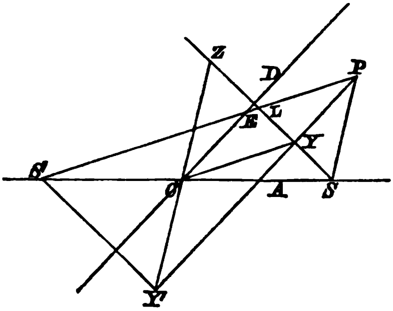
106. Prop. X. *The perpendiculars from the foci on any tangent meet the tangent on the auxiliary circle, and the semi-conjugate axis is a mean proportional between their lengths.*

Let SY, $S'Y'$ be the perpendiculars, and let SY produced meet $S'P$ in L.

Then the triangles SPY, LPY are equal in all respects,

and $SY = LY$.

Hence, C being the middle point of SS' and Y of SL, CY is parallel to $S'L$, and $S'L=2CY$.



But $$S'L=S'P-PL=S'P-SP=2AC;$$

$$\therefore\ CY=AC,$$

and Y is on the auxiliary circle.

So also Y' is a point in the circle.

Let SY produced meet the circle in Z, and join $Y'Z$; then, $Y'YZ$ being a right angle, ZY' is a diameter and passes through C. Hence, the triangles SCZ, $S'CY'$ being equal,

$$S'Y'=SZ,$$

and $$SY\,.\,S'Y'=SY\,.\,SZ=SA\,.\,SA'=BC^2.$$

Cor. 1. If P' be the other extremity of the diameter PC, the tangent at P' is parallel to PY, and therefore Z is the foot of the perpendicular from S on the tangent at P'.

Cor. 2. If the diameter DCD', drawn parallel to the tangent at P, meet $S'P$, SP in E and E', $PECY$ is a parallelogram;

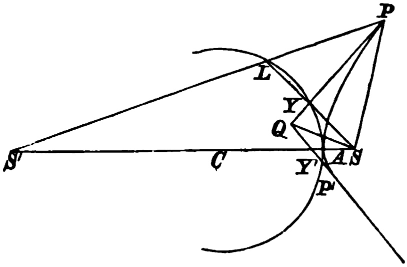
$$\therefore\ PE=CY=AC,$$

and so also $$PE'=CY'=AC.$$

107. Prop. XI. *To draw tangents to an hyperbola from a given point.*

The construction of Art. (14) may be employed, or, as in the cases of the ellipse and parabola, the following.

Let Q be the given point; join SQ, and upon SQ as diameter describe a circle intersecting the auxiliary circle in Y and Y';



QY and QY' are the required tangents.

Producing SY to L, for that $YL=SY$, draw $S'L$ cutting QY in P, and join SP.

The triangles SPY, LPY are equal in all respects,

and $$S'P-SP=S'L=2CY=2AC,$$

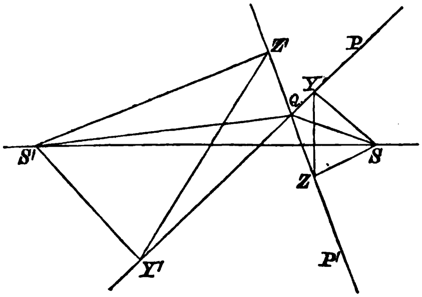
$\therefore$ P is a point on the hyperbola.

Also QP bisects the angle SPS', and is therefore the tangent at P. A similar construction will give the other tangent QP'.

If the point Q be within the angle formed by the asymptotes, the tangents will both touch the same branch of the curve; but if it lie within the external angle, they will touch opposite branches.

108. **Prop. XII.** *If two tangents be drawn from any point to an hyperbola they are equally inclined to the focal distances of that point.*

Let PQ, $P'Q$ be the tangents, SY, $S'Y'$, SZ, $S'Z'$ the perpendiculars from the foci; join YZ, $Y'Z'$.



Then the angles YSZ, $Y'S'Z'$ are equal, for they are the supplements of YQZ, $Y'QZ'$.

Also $SY . S'Y' = SZ . S'Z'$, Art. (106);

or $SY : SZ :: S'Z' : S'Y'$;

$\therefore$ the triangles YZS, $Y'S'Z'$ are similar,

and the angle $YZS = Z'Y'S'$.

But the angle $YQS = YZS$, and $Z'QS' = Y'Z'S'$;

$\therefore$ $YQS = Z'QS'$.

That is, the tangent QP, and the tangent $P'Q$ produced are equally inclined to SQ and $S'Q$.

Or, producing $S'Q$, QP and QP' are equally inclined to QS and $S'Q$ produced.

In exactly the same manner it can be shewn that if QP, QP' touch opposite branches of the curve the angles PQS, $P'QS'$ are equal.

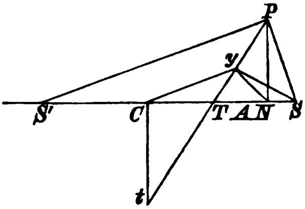
Cor. If Q be a point in a confocal hyperbola, the normal at Q bisects the angle between SQ and $S'Q$ produced, and therefore bisects the angle PQP'.

Hence, if from any point of an hyperbola tangents be drawn to a confocal hyperbola, these tangents are equally *inclined to the normal or the tangent at the point, according as it lies within or without that angle formed by the asymptotes of the confocal which contains the transverse axes.*

109. **Prop. XIII.** *If PT, the tangent at P, meet the transverse axis in T, and PN be the ordinate,*

$$CN \,.\, CT = AC^2.$$

Let fall the perpendicular Sy upon PT, and join yN, Cy, SP, and $S'P$.



The angle $CyT = S'Py$
$= SPy$
$=$ the supplement of SNy
$= CNy$;

also the angle yCT is common to the two triangles CyT, CyN; these triangles are therefore similar,

and $CN : Cy :: Cy : CT$,

or $CN \,.\, CT = Cy^2 = AC^2$.

Cor. 1. Hence $CN \,.\, NT = CN^2 - CN \,.\, CT$
$= CN^2 - AC^2$
$= AN \,.\, NA'$.

Cor. 2. Hence also it follows that

If any number of hyperbolas be described having the same transverse axis, and an ordinate be drawn cutting

the hyperbolas, the tangents at the points of section will all meet the transverse axis in the same point.

Cor. 3. If CN be increased indefinitely, CT is diminished indefinitely, and the tangent ultimately passes through C, as we have already shewn, Art. (100).

110. **Prop. XIV.** *If the tangent at P meet the conjugate axis in t, and PN be the ordinate,*

$$Ct \,.\, PN = BC.$$

For, Fig. Art. (109),

Then $\quad Ct : PN :: CT : NT;$

$$\therefore\ Ct\,.\,PN : PN^2 :: CT\,.\,CN : CN\,.\,NT,$$
$$:: AC^2 : AN\,.\,NA'.$$

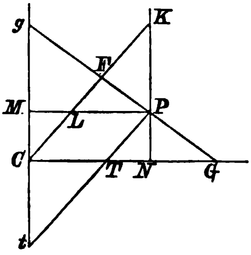
$$\therefore\ Ct\,.\,PN : AC^2 :: PN^2 : AN\,.\,NA',$$
$$:: BC^2 : AC^2,$$

and $\quad Ct\,.\,PN = BC^2.$

111. **Prop. XV.** *If the normal at P meet the transverse axis in G, the conjugate axis in g, and the diameter parallel to the tangent at P in F,*

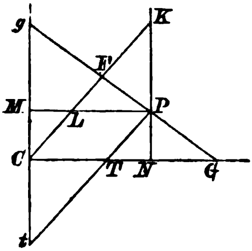
$$PF\,.\,PG = BC^2, \text{ and } PF\,.\,Pg = AC^2.$$

Let NP, PM, perpendicular to the axes, meet the diameter CF in K and L, and let the tangents meet the axes in T and t.



Then KNG, KFG being right angles, a circle can be described about $KFNG$, and therefore

$$PF \, . \, PG = PK \, . \, PN = Ct \, . \, PN$$
$$= BC^2.$$



Similarly a circle can be drawn through $FLMg$;

$$\therefore PF \, . \, Pg = PL \, . \, PM = CT \, . \, CN,$$
$$= AC^2.$$

Cor. Hence

$$PG : Pg :: BC^2 : AC^2.$$

Also, PNG, PMg are similar triangles;

$$\therefore NG : MP :: PG : Pg,$$
$$:: BC^2 : AC^2,$$

or $$NG : CN :: BC^2 : AC^2.$$

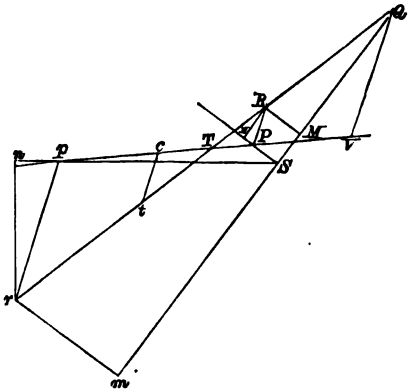
Hence $$CN : CG :: AC^2 : AC^2 + BC^2,$$
$$:: AC^2 : SC^2,$$

or $$CG : CN :: SC^2 : AC^2.$$

112. Prop. XVI. *If PCp be a diameter, and QV an ordinate, and if the tangent at Q meet the diameter Pp in T,*

$$CV \, . \, CT = CP^2.$$

Let the tangents at P and p meet the tangent at Q in R and r;



Then the angle $SPR = spr$, Cor. Art. (105), and therefore if RN, rn be the perpendiculars on SP, sp, the triangles RPN, rpn are similar.

Draw RM, rm perpendiculars on SQ.

Then $\qquad TR : Tr :: RP : rp,$

$$:: RN : rn,$$

$$:: RM : rm, \text{ Cor. Art. (13)},$$

$$:: RQ : rQ.$$

Hence, QV, RP, and rp being parallel,

$$TP : Tp :: PV : pV;$$

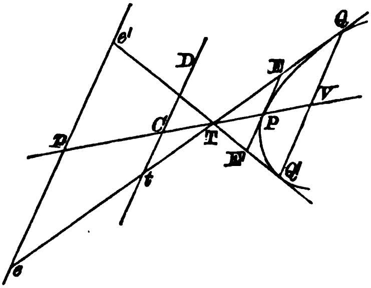
$$\therefore\ TP + Tp : Tp - TP :: PV + pV : pV - PV,$$

or
$$2CP : 2CT :: 2CV : 2CP,$$

or
$$CV \,.\, CT = CP^2.$$

113. Prop. XVII. *A diameter bisects all chords parallel to the tangents at its extremities.*

Let PCp be the diameter, and QQ' the chord, parallel



to the tangents at P and p. Then if the tangents TQ, TQ' at Q and Q' meet the tangents at P and p, in the points E, E', e, e',

$$EP = E'P \text{ and } ep = e'p, \text{ Art. (16);}$$

$\therefore$ The point T is on the line Pp;

but TP bisects QQ';

that is, the diameter pCP produced bisects QQ'.

Def. *The line DCd, drawn parallel to the tangent at P, and terminated by the conjugate hyperbola, that is, the diameter parallel to the tangent at P, is said to be conjugate to PCp.*

A diameter therefore bisects all chords parallel to its conjugate.

114. Prop. XVIII. *If the diameter DCd be conjugate to PCp, then will PCp be conjugate to DCd.*

Let the chord QVq be parallel to CD and be bisected in V by CP produced.

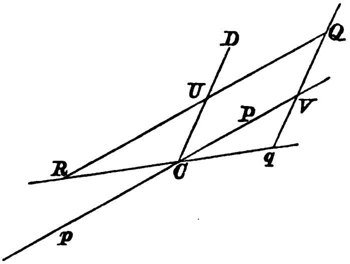
Draw the diameter qCR, and join RQ meeting CD in U.

Then $RC = Cq$ and $QV = Vq$;

$\therefore$ QR is parallel to CP.

Also $QU : UR :: Cq : CR$,

and $\therefore QU = UR$;



that is, CD bisects the chords parallel to CP, and PCp is therefore conjugate to DCd.

Hence when two diameters are conjugate, each bisects the chords parallel to the other.

115. Def. *Chords drawn from the extremities of any diameter to a point on the hyperbola are called supplemental chords.*

Thus, qQ, QR are supplemental chords, and they are parallel to CD and CP; supplemental chords are therefore parallel to conjugate diameters.

Def. *A line QV, drawn from any point Q of an hyperbola, parallel to a diameter DCd, and terminated by the conjugate diameter PCp, is called an ordinate of the diameter PCp, and if QV produced meet the curve in Q', QVQ' is the double ordinate.*

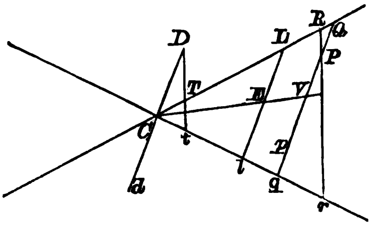
This definition includes the two cases in which QQ' may be drawn so as to meet the same, or opposite branches of the hyperbola.

Properties of Asymptotes.

116. Prop. XIX. *If from any point Q in an asymptote QPpq be drawn meeting the curve in P, p and the other asymptote in q, and if CD be the semi-diameter parallel to Qq,*

$$QP \,.\, Pq = CD^2 \textit{ and } QP = Pq.$$

Through P and D draw RPr, DTt perpendicular to the transverse axis, and meeting the asymptotes.



Then $QP : RP :: CD : DT$,

and $Pq : Pr :: CD : Dt$;

$\therefore QPPq : RP \,.\, Pr :: CD^2 : DT \,.\, Dt$.

But $RP \,.\, Pr = BC^2 = DT \,.\, Dt$, Art. (103),

$\therefore \; QP \,.\, Pq = CD^2$.

Similarly $qp \,.\, pQ = CD^2$;

$\therefore QP \,.\, Pq = qp \,.\, pQ$;

or, if V be the middle point of Qq,

$$QV^2 - PV^2 = QV^2 - pV^2.$$

Hence $PV = pV$,

and $\therefore PQ = pq$.

We have taken the case in which Qq meets one branch of the hyperbola. It may however be shewn in the same manner that the same relations hold good for the case in which Qq meets opposite branches.

Cor. *If a straight line $PP'p'p$ meet the hyperbola in P, p, and the conjugate hyperbola in P', p', $PP' = pp'$.*

For, if the line meet the asymptotes in Q, q,

$$QP' = p'q,$$

and $$PQ = qp\,;$$

$$\therefore\ PP' = pp'.$$

117. Prop. XX. *The portion of a tangent, which is terminated by the asymptotes, is bisected at the point of contact, and is equal to the parallel diameter.*

LEl being the tangent, Fig. Art. (116), and DCd the parallel diameter, draw any parallel straight line $QPpq$ meeting the curve and the asymptotes.

Then $QP = pq$; and, if the line move parallel to itself until it coincides with Ll, the points P and p coincide with E, and $\therefore$ $LE = El$.

Also $QP\,.\,Pq = CD^2$, always, and therefore

$$LE\,.\,El = CD^2,$$

or $LE = CD$.

It may be noticed that since the asymptotes are tangents, the fact that $LE = El$ is a particular case of the general property demonstrated in Art. (16).

Properties of Conjugate Diameters.

118. Prop. XXI. *Conjugate diameters of an hyperbola are also conjugate diameters of the conjugate hyperbola, and the asymptotes are diagonals of the parallelogram formed by the tangents at their extremities.*

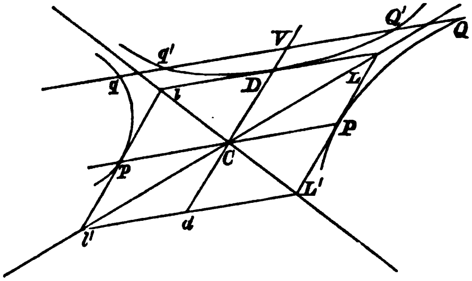
PCp and DCd being conjugate, let QVq, a double ordinate of CD, meet the conjugate hyperbola in Q' and q'.

Then $$QV = Vq,$$

and $$QQ' = qq', \text{ Cor. Art. (116)};$$

$$\therefore\ Q'V = Vq'.$$

That is, CD bisects the chords of the conjugate hyperbola parallel to CP.



Hence CD and CP are conjugate in both hyperbolas, and therefore the tangent at D is parallel to CP.

Let the tangent at P meet the asymptote in L; then

$$PL = CD. \quad \text{Art. (117).}$$

Hence LD is parallel and equal to CP;

but the tangent at D is parallel to CP;

$\therefore$ LD is the tangent at D.

Completing the figure, the tangents at p and d are parallel to those at P and D, and therefore the asymptotes are the diagonals of the parallelogram $Ll'L'$.

Cor. Hence, joining PD, it follows that PD is parallel to the asymptote lCL', since $LP = PL'$, and $LD = Dl$.

119. Prop. XXII. *If QV be an ordinate of a diameter PCp, and DCd the conjugate diameter,*

$$QV^2 : PV \,.\, Vp :: CD^2 : CP^2.$$

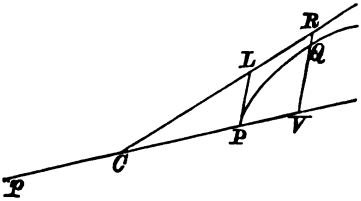
Let QV, and the tangent at P, meet the asymptote in R and L.

Then LP being equal to CD,

$$RV^2 : CD^2 :: CV^2 :: CP^2;$$

$$\therefore RV^2 - CD^2 : CD^2 :: CV^2 - CP^2 : CP^2.$$

But $$RV^2 - QV^2 = CD^2.$$



Hence $QV^2 : CD^2 :: CV^2 - CP^2 : CP^2$,

or $QV^2 : PV \,.\, Vp :: CD^2 : CP^2.$

120. **Prop. XXIII.** *If QV be an ordinate of a diameter PCp, and if the tangent at Q meet the conjugate diameter, DCd, in t,*

$$Ct \,.\, QV = CD^2.$$

For, Fig. Art. (113),

$$Ct : QV :: CT : VT,$$

and $\therefore Ct \,.\, QV : QV^2 :: CV \,.\, CT : CV \,.\, VT.$

But $CV \,.\, CT = CP^2$, Art. (112),

and $CV \,.\, VT = CV^2 - CV \,.\, CT = CV^2 - CP^2$;

$$\therefore Ct \,.\, QV : QV^2 :: CP^2 : CV^2 - CP^2,$$
$$:: CD^2 : QV^2. \text{ Art. (119).}$$

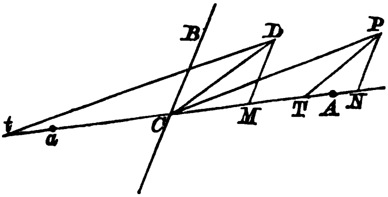
Hence $$Ct \,.\, QV = CD^2.$$

121. **Prop. XXIV.** *If ACa, BCb be conjugate diameters, and PCp, DCd another pair of conjugate diameters, and if PN, DM be ordinates of ACa,*

$$CM : PN :: AC : BC,$$

and $$DM : CN :: BC : AC.$$

Let the tangents at P and D meet ACa in T and t;



Then $CN.CT=AC^2=CM.Ct$, Arts. (112) and (120),

$$\begin{aligned}\therefore\ CM : CN &:: CT : Ct,\\ &:: PT : CD,\\ &:: PN : DM,\\ &:: CN : Mt;\end{aligned}$$

$$\therefore\ CN^2=CM.Mt=CM^2+CM.Ct$$
$$=CM^2+AC^2,$$

and $$CM^2=CN^2-AC^2.$$

But $$PN^2 : CN^2-AC^2 :: BC^2 : AC^2;$$

$$\therefore\ PN : CM :: BC : AC;$$

and, similarly, $$DM : CN :: BC : AC.$$

Cor. We have shewn in the course of the proof, that

$$CN^2-CM^2=AC^2.$$

Similarly, if Pn, Dm be ordinates of BC,

$$Cm^2-Cn^2=BC^2;$$

that is, $$DM^2-PN^2=BC^2;$$

and it must be noticed that these relations are shewn for any pair of conjugate diameters ACa, BCb, including of course the axes.

122. **Prop. XXV.** *If CP, CD be conjugate semi-diameters, and AC, BC the semi-axes,*

$$CP^2 - CD^2 = AC^2 - BC^2.$$

For, drawing the ordinates PN, DM, and, remembering that in this case the angles at N and M are right angles, we have, from the figure of the previous article,

$$CP^2 = CN^2 + PN^2,$$

$$CD^2 = CM^2 + DM^2.$$

But $CN^2 - CM^2 = AC^2$ and $DM^2 - PN^2 = BC^2$;

$$\therefore\ CP^2 - CD^2 = AC^2 - BC^2.$$

123. **Prop. XXVI.** *If the normal at P meet the axes in G and g,*

$$PG : CD :: BC : AC,$$

and $$Pg : CD :: AC : BC.$$

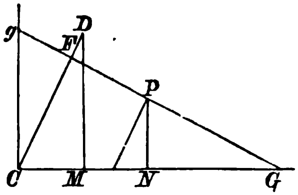
For the proofs of these relations, see Art. (79).

Observe also that

$$PG \,.\, Pg = CD^2.$$

124. **Prop. XXVII.** *The area of the parallelogram formed by the tangents at the ends of conjugate diameters is equal to the rectangle contained by the axes.*

Let CP, CD be the semi-diameters, and PN, DM the ordinates of the transverse axis.



Let the normal at P meet CD in F, and the axis in G. Then PNG, CDM are similar triangles, and, exactly as in Art. (80), it can be shewn that

$$PF \, . \, CD = AC \, . \, BC.$$

125. Prop. XXVIII. *If SP, $S'P$ be the focal distance of a point P, and CD be conjugate to CP,*

$$SP \, . \, S'P = CD^2.$$

Attending to the figure of Art. (106), the proof is the same as that of Art. (81).

126. Prop. XXIX. *If the tangent at P meet a pair of conjugate diameters in T and t, and CD be conjugate to CP,*

$$PT \, . \, Pt = CD^2.$$

This can be proved as in Art. (82).

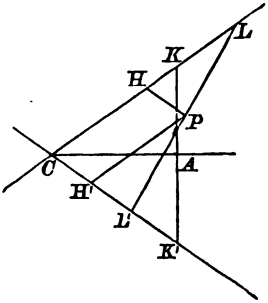
It can also be shewn that if the tangent at P meet two parallel tangents in T' and t',

$$PT' \, . \, Pt' = CD^2.$$

127. Prop. XXX. *If the tangent at P meet the asymptotes in L and L',*

$$CL \, . \, CL' = SC^2.$$

Let the tangent at A meet the asymptotes in K and



K'; then, Art. 124, the triangles LCL', KCK' are of equal area, and therefore

$$CL : CK' :: CK : CL', \text{ Euclid, Book VI.,}$$

or

$$CL . CL' = CK^2$$
$$= AC^2 + BC^2 = SC^2.$$

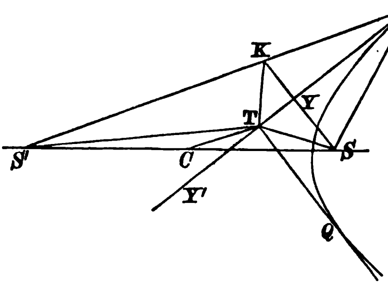
COR. If PH, PH' be drawn parallel to, and terminated by the asymptotes,

$$4 . PH . PH' = CS^2,$$

for $CL = 2PH'$, and $CL' = 2PH$.

128. PROP. XXXI. *Pairs of tangents at right angles to each other intersect on a fixed circle.*

PT, QT being two tangents at right angles, let SY, perpendicular to PT, meet $S'P$ in K.



Then, Art. (108), the angle

$$S'TY' = QTS,$$

and obviously, $$KTP = PTS;$$

therefore $S'TY'$ is complementary to KTP, and $S'TK$ is a right angle.

Hence,

$$4AC^2=S'K^2=S'T^2+TK^2$$
$$=S'T^2+ST^2$$
$$=2\,.\,CT^2+2\,.\,CS^2 \text{ by Euclid II. 12 and 13;}$$
$$\therefore\ CT^2=AC^2-BC^2,$$

and the locus of T is a circle.

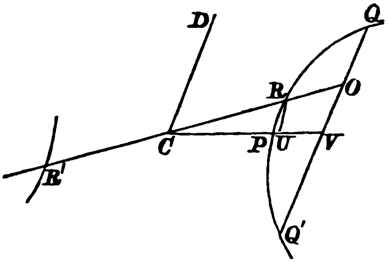
If AC be less than BC, this relation is impossible.

In this case, however, the angle between the asymptotes is greater than a right angle, and the angle PTQ between a pair of tangents being always greater than the angle between the asymptotes is greater than a right angle. The problem is therefore *à priori* impossible for the hyperbola, but becomes possible for the conjugate hyperbola.

As in the case of the ellipse, the locus of T is called the director circle.

129. Prop. XXXII. *The rectangles contained by the segments of any two chords which intersect each other are in the ratio of the squares of the parallel diameters.*

Through any point O in a chord QOQ' draw the diameter ORR'; and let CD be parallel to QQ', CP conjugate D, and bisecting QQ' in V.



Draw RU an ordinate of CP.

Then $$CD^2+RU^2 : CU^2 :: CD^2 : CP^2,$$
$$:: CD^2+QV^2 : CV^2.$$

But $$RU^2 : CU^2 :: OV^2 : CV^2;$$

$$\therefore\ CD^2 : CU^2 :: CD^2+QV^2-OV^2 : CV^2,$$

or $$CD^2 : CD^2+QV^2-OV^2 :: CU^2 : CV^2,$$
$$:: CR^2 : CO^2;$$

$$\therefore\ CD^2 : QV^2-OV^2 :: CR^2 : CO^2-CR^2,$$

or $$CD^2 : QO.OQ' :: CR^2 : OR.OR'.$$

Similarly, if qOq' be any other chord, and Cd the parallel semi-diameter,

$$Cd^2 : qO.Oq' :: CR^2 : OR.OR';$$

$$\therefore\ QO.OQ' : qO.Oq' :: CD^2 : Cd^2;$$

that is, the ratio of the rectangles depends only on the directions of the chords.

130. Prop. XXXIII. *If a circle intersect an hyperbola in four points, the several pairs of the chords of intersection are equally inclined to the axes.*

For the proof, see Art. (87).

Examples.

1. If a circle be drawn so as to touch two fixed circles externally, the locus of its centre is an hyperbola.

2. If the tangent at B to the conjugate meet the latus rectum in D, the triangles SCD, SXD are similar.

3. The straight line drawn from the focus to the directrix, parallel to an asymptote, is equal to the semi-latus-rectum, and is bisected by the curve.

4. Given the asymptotes and a focus, find the directrix.

5. Given the centre, one asymptote, and a directrix, find the focus.

6. Parabolas are described passing through two fixed points, and having their axes parallel to a fixed line; the locus of their foci is an hyperbola.

7. The base of a triangle being given, and also the point of contact with the base of the inscribed circle, the locus of the vertex is an hyperbola.

8. If the normal at P meet the conjugate axis in g, and gN be the perpendicular on SP, then $PN=AC$.

9. Draw a tangent to an hyperbola, or its conjugate, parallel to a given line.

10. If AA' be the axis of an ellipse, and PNP' a double ordinate, the locus of the intersection of $A'P$ and $P'A$ is an hyperbola.

11. The tangent at P bisects any straight line perpendicular to AA', and terminated by AP, and $A'P$.

12. If PCp be a diameter, and if Sp meet the tangent at P in T,

$$SP=ST.$$

13. Given an asymptote, the focus, and a point; construct the hyperbola.

14. A circle can be drawn through the foci and the intersections of any tangent with the tangents at the vertices.

15. Given an asymptote, the directrix, and a point; construct the hyperbola.

16. If through any point P of an hyperbola straight lines are drawn parallel to the asymptotes and meeting any semidiameter CQ in P and R,

$$CP\ .\ CR=CQ^2.$$

17. PN is an ordinate and NQ parallel to AB meets the conjugate axis in Q; prove that $QB\ .\ QB'=PN^2$.

18. NP is an ordinate and Q a point in the curve; AQ, $A'Q$ meet NP in D and E; prove that $ND\ .\ NE=NP^2$.

19. If a tangent cut the major axis in the point T, and perpendiculars $SY.HZ$ be let fall on it from the foci, then $AT.A'T = YT.ZT$.

20. In the tangent at P a point Q is taken such that PQ is proportional to CD; shew that the locus of Q is an hyperbola.

21. Tangents are drawn to an hyperbola, and the portion of each tangent intercepted by the asymptotes is divided in a constant ratio; prove that the locus of the point of section is an hyperbola.

22. If the tangent and normal at P meet the conjugate axis in t and K respectively, prove that a circle can be drawn through the foci and the three points P, t, K.

Shew also that

$$GK : SK :: SA : AX,$$

and

$$St : tK :: BC : CD,$$

CD being conjugate to CP.

23. Shew that the points of trisection of a series of conterminous circular arcs lie on branches of two hyperbolas; and determine the distance between their centres.

24. If the tangent at any point P of an hyperbola cut an asymptote in T, and if SP cut the same asymptote in Q, then $SQ=QT$.

25. A series of hyperbolas having the same asymptotes is cut by a straight line parallel to one of the asymptotes, and through the points of intersection lines are drawn parallel to the other, and equal to either semi-axis of the corresponding hyperbola, prove that the locus of their extremities is a parabola.

26. Prove that the rectangle $PY.PY'$ in an ellipse is equal to the conjugate axis of the confocal hyperbola passing through P.

27. If the tangent at P meet one asymptote in T, and a line TQ be drawn parallel to the other asymptote to meet the curve in Q; prove that if PQ be joined and produced both ways to meet the asymptotes in R and R', RR' will be trisected at the points P and Q.

28. The tangent at a point P of an ellipse meets the hyperbola having the same axes as the ellipse in C and D. If Q be

the middle point of CD, prove that OQ and OP are equally inclined to the axes, O being the centre of the ellipse.

29. Given one asymptote, the direction of the other, and the position of one focus, determine the position of the vertices.

30. Two points of an hyperbola are taken on the same branch of the curve, and on the same side of the axis; prove that a circle can be drawn touching the four focal distances.

31. In an hyperbola, supposing the two asymptotes and one point of the curve to be given in position, shew how to construct the curve; and find the position of the foci.

32. Given a pair of conjugate diameters, construct the axes.

33. If PH, PK be drawn parallel to the asymptotes from a point P on the curve, and if a line through the centre meet them in R, T, and the parallelogram $PRQT$ be completed, Q is a point on the curve.

34. The rectangles contained by the segments of focal chords are in the ratio of the chords.

35. If a given point be the focus of any hyperbola, passing through a given point and touching a given straight line, prove that the locus of the other focus is an arc of a fixed hyperbola.

36. An ellipse and hyperbola are described, so that the foci of each are at the extremities of the transverse axis of the other; prove that the tangents at their points of intersection meet the conjugate axis in points equidistant from the centre.

37. A circle is described about the focus as centre, with a radius equal to one-fourth of the latus rectum: prove that the focal distances of the points at which it intersects the hyperbola are parallel to the asymptotes.

38. The tangent at any point on an hyperbola is produced to meet the asymptotes, thus forming a triangle: determine the locus of the point of intersection of the straight lines drawn from the angles of this triangle to bisect the opposite sides.

39. If SY, $S'Y'$ be the perpendiculars on the tangent at P, a circle can be drawn through the points, Y, Y', N, C.

40. The straight lines joining each focus to the foot of the perpendicular from the other focus on the tangent meet on the normal and bisect it.

41. If the tangent and normal at P meet the axis in T and G, $NG . CT = BC^2$.

42. If the tangent at P meet the axis in T and t, the angles PSt, STP are supplementary.

43. If the tangent at P meet any conjugate diameters in T and t, the triangles SPT, $S'Pt$ are similar.

44. If the diameter conjugate to CP meet SP and $S'P$ in E and E', prove that the circles about the triangles SCE, $S'CE'$ are equal.

45. The locus of the centre of the circle inscribed in the triangle SPS' is a straight line.

46. If PN be an ordinate, and NQ parallel to AP meet CP in Q, AQ is parallel to the tangent at P.

47. If an asymptote meet the directrix in D, and the tangent at the vertex in E, AD is parallel to SE.

48. The radius of the circle touching the curve and its asymptotes is equal to the portion of the latus rectum produced, between its extremity and the asymptote.

49. If G be the foot of the normal, and if the tangent meet the asymptotes in L and M, $GL = GM$.

50. With two conjugate diameters of an ellipse as asymptotes, a pair of conjugate hyperbolas is constructed: prove that if one hyperbola touch the ellipse, the other will do so likewise; prove also that the diameters drawn through the points of contact are conjugate to each other.

51. If two tangents be drawn to an hyperbola the lines joining their intersections with the asymptotes will be parallel.

52. The locus of the centre of the circle touching SP, HP produced, and the major axis, is an hyperbola.

53. If from a point P in an hyperbola, PK be drawn parallel to an asymptote to meet the directrix in K, then $PK = SP$.

54. If PD be drawn parallel to an asymptote, to meet the conjugate hyperbola in D, CP and CD are conjugate diameters.

55. If QR be a chord parallel to the tangent at P, and if QL, PN, RM be drawn parallel to one asymptote to meet the other,

$$CL . CM = CN^2.$$

56. If a circle touch the transverse axis at a focus, and pass through one end of the conjugate, the chord intercepted by the conjugate is a third proportional to the conjugate and transverse semi-axes.

57. A line through one of the vertices, terminated by two lines drawn through the other vertex parallel to the asymptotes is bisected at the other point where it cuts the curve.

58. If PSQ be a focal chord, and if the tangents at P and Q meet in T, the difference between PTQ and half $PS'Q$ is a right angle.

59. If a straight line passing through a fixed point C meet two fixed lines OA, OB in A and B, and if P be taken in AB such that $CP^2 = CA \cdot CB$, the locus of P is an hyperbola, having its asymptotes parallel to OA, OB.

60. If from the points P and Q in an hyperbola there be drawn PL, QM, parallel to each other to meet one asymptote, and PR, QN also parallel to each other to meet the other asymptote, $PL \cdot PR = QM \cdot QN$.

61. Any diameter is a mean proportional between the parallel focal chord and the transverse axis.

62. An ordinate VQ of any diameter CP is produced to meet the asymptote in R, and the conjugate hyperbola in Q'; prove that $QV^2 + Q'V^2 = 2RV^2$.

Prove also that the tangents at Q and Q' meet the diameter CP in points equidistant from C.

63. A chord QPL meets an asymptote in L, and a tangent from L is drawn touching at R; if PM, RE, QN, be drawn parallel to the asymptote to meet the other,

$$PM + QN = 2 \cdot RE.$$

64. Tangents are drawn from any point in a circle through the foci; prove that the lines bisecting the angle between the tangents, or between one tangent and the other produced, all pass through a fixed point.

65. If a circle through the foci meet two confocal hyperbolas in P and Q, the angle between the tangents at P and Q is equal to PSQ.

66. If SY, $S'Y'$ be perpendiculars on the tangent at P, and if PN be the ordinate, the angles PNY, PNY' are supplementary.

67. Find the position of P when the area of the triangle YCY' is the greatest possible, and shew that, in that case,

$$PN \,.\, SC = BC^2.$$

68. If the tangent at P meet the conjugate axis in t, the areas of the triangles SPS', StS' are in the ratio of $CD^2 : St^2$.

69. If SY, SZ be perpendiculars on two tangents which meet in T, YZ is perpendicular to $S'T$.

70. A circle passing through a focus, and having its centre on the transverse axis touches the curve; shew that the focal distance of the point of contact is equal to the Latus Rectum.

71. If CQ be conjugate to the normal at P, then is CP conjugate to the normal at Q.

72. From a point in the auxiliary circle lines are drawn touching the curve in P and P'; prove that SP, $S'P'$ are parallel.

73. If the tangent and normal at P meet the axis in T and G,

$$CT \,.\, CG = SC^2.$$

74. Find the locus of the points of contact of tangents to a series of confocal hyperbolas from a fixed point in the axis.

75. Tangents to an hyperbola are drawn from any point in one of the branches of the conjugate, shew that the chord of contact will touch the other branch of the conjugate.

76. An ordinate NP meets the conjugate hyperbola in Q; prove that the normals at P and Q meet on the transverse axis.

77. A parabola and an hyperbola have a common focus S and their axes in the same direction. If a line SPQ cut the curves in P and Q, the angle between the tangents at P and Q is equal to half the angle between the axis and the other focal distance of the hyperbola.

78. If a hyperbola be described touching the four sides of a quadrilateral which is inscribed in a circle, and one focus lie on the circle, the other focus will also lie on the circle.

79. A conic section is drawn touching the asymptotes of an hyperbola. Prove that two of the chords of intersection of the curves are parallel to the chord of contact of the conic with the asymptotes.

80. A parabola P and an hyperbola H have a common focus, and the asymptotes of H are tangents to P; prove that the tangent at the vertex of P is a directrix of H, and that the tangent to P at the point of intersection passes through the further vertex of H.

81. From any point T, in the tangent at P, TQ is drawn perpendicular to SP, and TR perpendicular to the transverse axis meets the curve in R; prove that $SQ = QR$.

82. If an ellipse and a confocal hyperbola intersect in P, an asymptote passes through the point on the auxiliary circle of the ellipse corresponding to P.

83. P is a point on an hyperbola whose foci are S and H; another hyperbola is described whose foci are S and P, and whose transverse axis is equal to $SP - 2PH$: shew that the hyperbolas will meet only at one point, and that they will have the same tangent at that point.

84. A point D is taken on the axis of an hyperbola, of which the excentricity is 2, such that its distance from the focus S is equal to the distance of S from the further vertex A'; P being any point on the curve, $A'P$ meets the latus rectum in K. Prove that DK and SP intersect on a certain fixed circle.

85. Shew that the locus of the point of intersection of tangents to a parabola making with each other a constant angle equal to half a right angle, is an hyperbola.

86. The tangent and normal at any point intersect the asymptotes and axes respectively in four points which lie on a circle passing through the centre of the curve.

The radius of this circle varies inversely as the perpendicular from the centre on the tangent.

87. The difference between the sum of the squares of the distances of any point from the ends of any diameter and the sum of the squares of its distances from the ends of the conjugate is constant.

88. If a tangent meet the asymptotes in L and M, the angle subtended by LM at the farther focus is half the angle between the asymptotes.

89. If PN be the ordinate of P, and PT the tangent, prove that

$$SP : ST :: AN : AT.$$

90. PCP' is a given diameter of an ellipse, QVQ' a double ordinate to PP'; the locus of the intersection of PQ', $P'Q$ is an hyperbola.

91. Two sides of a quadrilateral are given in magnitude and position; if the quadrilateral be such that a circle can be inscribed in it, the locus of the point of intersection of the other two sides is an hyperbola.

92. A tangent to an hyperbola at P meets the conjugate axis in t, and tQ is perpendicular to SP; prove that SQ is of constant length.

93. An hyperbola, having a given transverse axis, has one focus fixed, and always touches a given straight line; the locus of the other focus is a circle.

94. A chord $PRVQ$ meets the directrices in R and V; shew that PR and VQ subtend, each at the focus nearer to it, angles of which the sum is equal to the angle between the tangents at P and Q.

95. The points, in which the tangents at the extremities of the transverse axis of an ellipse are cut by the tangent at any point of the curve, are joined, one with each focus; prove that the point of intersection of the joining lines lies in the normal at the point.

96. A parabola is described touching the conjugate axes of a hyperbola at their extremities; prove that one asymptote is parallel to the axis of the parabola, and that the other asymptote is parallel to the chords of the parabola bisected by the first.

If a straight line parallel to the second asymptote meet the hyperbola and its conjugate in P, P', and the parabola in Q, Q', it may be shewn that $PQ = P'Q'$.

97. If two points E and E' be taken in the normal PG such that $PE=PE'=CD$, the loci of E and E' are hyperbolas having their axes equal to the sum and difference of the axes of the given hyperbola.

98. The angular point A of a triangle ABC is fixed, and the angle A is given, while the points B and C move on a fixed straight line; prove that the locus of the centre of the circle circumscribing the triangle is a hyperbola, and that the envelope of the circle is another circle.

99. If a conic be described having for its axes the tangent and normal at any point of a given ellipse, and touching at its centre the axis-major of the given ellipse, and if another conic be described in the same manner but touching the minor axis at the centre, prove that the foci of these conics lie in two circles concentric with the given ellipse, and having their diameters equal to the sum and difference of its axes.

100. An ellipse and an hyperbola are confocal; if a tangent to one intersect at right angles a tangent to the other, the locus of the point of intersection is a circle.

Shew also that the difference of the squares on the distances from the centre of parallel tangents is constant.

CHAPTER V.

THE RECTANGULAR HYPERBOLA.

If the axes of an hyperbola be equal, the angle between the asymptotes is a right angle, and the curve is called equilateral or rectangular.

131. Prop. I. *In a rectangular hyperbola*

$$CS^2=2AC^2, \textit{ and } SA^2=2AX^2.$$

For $$CS^2=AC^2+BC^2=2AC^2,$$

and $$SA : AX :: SC : AC;$$

$$\therefore SA^2=2AX^2.$$

Observe that in the figure of Art. (98), SDC is an isosceles triangle, since

$$SD=BC, \text{ and } CD=AC,$$

and therefore $$SD=DC.$$

132. Prop. II. *The asymptotes of a rectangular hyperbola bisect the angles between any pair of conjugate diameters.*

For, in a rectangular or equilateral hyperbola,

$$CA=CB,$$

and therefore, since $CP^2-CD^2=CA^2-CB^2$,

$$CP=CD,$$

CP, CD being any conjugate semi-diameters.

Also, figure Art. (118), the parallelogram $CPLD$ is a rhombus, and therefore CL bisects the angle PCD.

COR. Supplemental chords are equally inclined to the asymptotes, for they are parallel to conjugate diameters.

133. PROP. III. *Diameters at right angles to each other are equal.*

Let CP, CP' be semi-diameters at right angles to each other, and CD conjugate to CP.

Then, if CL, CL' be the asymptotes, the angle

$$\begin{aligned} P'CL' &= PCL \\ &= DCL; \end{aligned}$$

$$\therefore CP' = CD = CP.$$

134. PROP. IV. *If the normal at P meet the axes in G and g,*

$$CN = NG \text{ and } PG = Pg = CD,$$

CD being conjugate to CP.

For, Art. (111),

$$NG : CN :: BC^2 : AC^2;$$

$$\therefore NG = CN.$$

Also $$PF \,.\, PG = BC^2 \text{ and } PF \,.\, Pg = AC^2;$$

$$\therefore PG = Pg.$$

Further, Art. (123),

$$PG : CD :: BC : AC;$$

$$\therefore PG = CD.$$

135. PROP. V. *If QV be an ordinate of a diameter PCp,*

$$QV^2 = PV \,.\, Vp.$$

For $$QV^2 : PV \,.\, Vp :: CD^2 : CP^2,$$

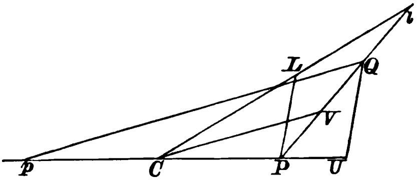
and $$CD = CP;$$

$$\therefore QV^2 = PV \,.\, Vp = CV^2 - CP^2.$$

136. **Prop. VI.** *If PQ be a chord, and CV the diameter conjugate to PQ, the angle between PQ and the tangent at P is equal to the angle VCP.*

Let PQ and the tangent at P meet the asymptote in l and L. Then the angle

$$\begin{aligned} LPQ &= PLC - VlC \\ &= LCP - VCl \\ &= VCP. \end{aligned}$$



Hence, if P be the other end of the diameter PC, the angle

$$QpP = VCP = LPQ.$$

Or thus, let QU parallel to the tangent at P, meet CP produced in U.

Then $$QU^2 = PU \,.\, Up,$$

or, $$QU : PU :: Up : UQ.$$

Therefore the triangles PQU, QUp are similar, and the angle $QpU = PQU = LPQ$.

137. **Prop. VII.** *Any chord subtends, at the ends of any diameter, angles which are equal or supplementary.*

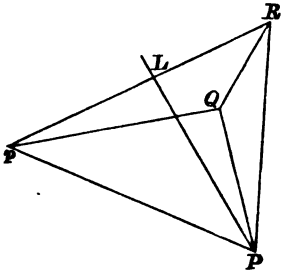
This theorem divides itself into four cases, which are shewn in the appended figures.

Let QR be the chord, and Pp the diameter. Then, if LP be the tangent at P, fig. (1), the angle

$$LPQ = QpP,$$

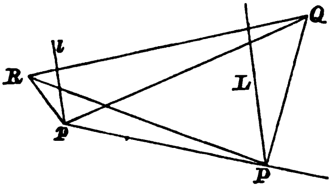
and $$LPR = RpP;$$

$$\therefore\ QPR = QpR.$$



In fig. (2), if pl be the tangent at p, parallel to PL,

$$QpR = Qpl + lpR = Qpl + pPR,$$



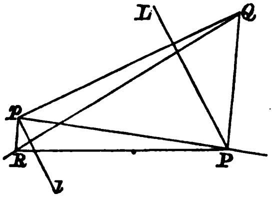
and $$QPR = QPL + LPR = QpP + LPR;$$

$$\therefore\ QpR + QPR = LpP + LPp,$$

that is, QpR and QPR are together equal to two right angles.

In fig. (3)

$$\begin{aligned} QPR &= QPL + LPp + pPR \\ &= QpP + Ppl + lpR \\ &= QpR. \end{aligned}$$



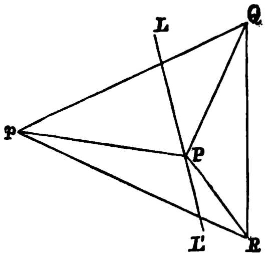
In fig. (4)

$$QPL = QpP,$$

and

$$RPL' = RpP\,;$$

$$\therefore\ QpR = QPL + RPL'\,;$$



therefore QpR and QPR are together equal to two right angles.

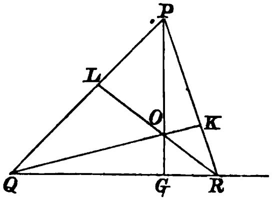
Hence it will be seen that when QR, or QR produced, meet the diameter Pp between P and p, the angles subtended at P and p are equal; in other cases they are supplementary.

138. Prop. VIII. *If a rectangular hyperbola circumscribe a triangle, it passes through the orthocentre.*

Note. *The orthocentre is the point of intersection of the perpendiculars from the angular points on the opposite sides.*

If O be the orthocentre, the triangles LOP, LQR are similar, and

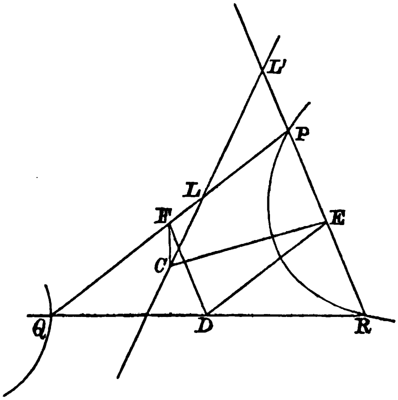
$$LO : LP :: LQ : LR;$$
$$\therefore LO . LR = LP . LQ.$$



But, if a rectangular hyperbola pass through P, Q, R, the diameters parallel to LR, PQ are equal: hence O is a point on the curve.

139. PROP. IX. *If a rectangular hyperbola circumscribe a triangle, the locus of its centre is the nine-point circle of the triangle.*

If PQR be the triangle, let L, L' be the points in which an asymptote meets the sides PQ, PR.



Join C, the centre of the hyperbola, with E and F, the middle points of PR and PQ.

Then CF is conjugate to PQ, and CE to PR; therefore the angle

$$\begin{aligned} FCE &= FCL + L'CE \\ &= CLF + EL'C \\ &= PLL' + PL'L \\ &= FPE \\ &= FDE, \end{aligned}$$

if D be the middle point of QR.

The circle passing through D, E, F therefore passes through C; that is, C lies on the nine-point circle.

A similar proof is applicable to the case in which the points P, Q, R lie on the same branch of the hyperbola.

140. The proof of Prop. (xxxiii), Art. (130), appears to fail in this case, since it does not follow, when two diameters are equal that they are equally inclined to the axes.

It is however obvious that, if a circle intersect an hyperbola, either the four points of intersection are all on one branch of the curve, or there are two on each branch; on the other hand, of two conjugate diameters, or of two diameters at right angles, one meets the curve and the other does not. Hence if chords be drawn parallel to these diameters one chord will meet opposite branches, and the other will meet one branch only; the cases are therefore distinct, and the proof holds good.

EXAMPLES.

1. A circle is described on the transverse axis as diameter. Prove that if any tangent be drawn to the hyperbola, the straight lines joining the centre of the hyperbola with the point of contact and with the middle point of the chord of intersection of the tangent with the circle, are inclined to the asymptotes at complementary angles.

2. PCP is a transverse diameter, and QV an ordinate; shew that QV is the tangent at Q to the circle circumscribing the triangle PQp.

3. If the tangent at P meet the asymptotes in L and M, and the normal meet the transverse axis in G, a circle can be drawn through C, L, M, and G, and LGM is a right angle.

4. Find the locus of the middle point of a line cutting off a constant area from the corner of a square.

5. If AA' be any diameter of a circle, PP' any ordinate to it, then the locus of the intersections of AP, $A'P'$ is a rectangular hyperbola.

6. If from the extremities of any diameter lines be drawn to any point in the curve, they will be equally inclined to the asymptotes.

7. Given an asymptote and a tangent at a given point, construct a rectangular hyperbola.

8. If CP, CD, and CP', CD' be two pairs of conjugate semi-diameters, prove that the angles PCP', DCD' are equal.

9. Focal chords parallel to conjugate diameters are equal.

10. If CP, CD be conjugate semi-diameters, and PN, DM ordinates of any diameter, the triangles PCN, DCM are equal in all respects.

11. The distance of any point from the centre is a geometric mean between its distances from the foci.

12. If P be a point on an equilateral hyperbola, and if the tangent at Q meet CP in T, the circle circumscribing CTQ touches the ordinate QV conjugate to CP.

13. If a circle be described on SS' as diameter, the tangents at the vertices will intersect the asymptotes in the circumference.

14. If two concentric rectangular hyperbolas be described, the axes of one being the asymptotes of the other, they will intersect at right angles.

15. If the tangents at two points Q and Q' meet in T, and if CQ, CQ' meet these tangents in R and R', the circle circumscribing RTR' passes through C.

16. If from a point Q in the conjugate axis QA be drawn to the vertex, and QR parallel to the transverse axis to meet the curve, $QR=AQ$.

17. Straight lines, passing through a given point, are bounded by two fixed lines at right angles to each other; find the locus of their middle points.

18. Given a point Q and a straight line AB, if a line QCP be drawn cutting AB in C, and P be taken in it, so that PD being a perpendicular upon AB, CD may be of constant magnitude, the locus of P is a rectangular hyperbola.

19. Every conic passing through the centres of the four circles which touch the sides of a triangle, is a rectangular hyperbola.

20. Ellipses are inscribed in a given parallelogram, shew that their foci lie on a rectangular hyperbola.

21. If two focal chords be parallel to conjugate diameters, the lines joining their extremities intersect on the asymptotes.

22. If P, Q be two points of a rectangular hyperbola, centre O, and QN the perpendicular let fall on the tangent at P, the circle through O, N, and P will pass through the middle point of the chord P, Q.

Having given the centre, a tangent, and a point of a rectangular hyperbola, construct the asymptotes.

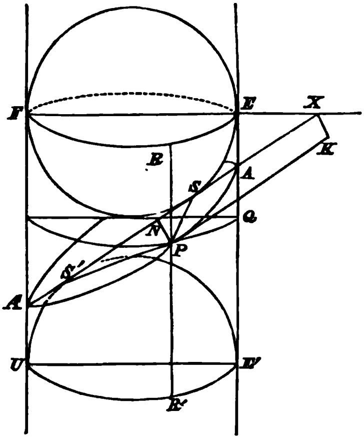
23. If a right-angled triangle be inscribed in the curve, the normal at the right angle is parallel to the hypothenuse.

24. On opposite sides of any chord of a rectangular hyperbola are described equal segments of circles; shew that the four points, in which the circles, to which these segments belong, again meet the hyperbola, are the angular points of a parallelogram.

25. Two lines of given lengths coincide with and move along two fixed lines, in such a manner that a circle can always be drawn through their extremities; the locus of the centre is a rectangular hyperbola.

26. If a rectangular hyperbola, having its asymptotes coincident with the axes of an ellipse, touch the ellipse, the axis of the hyperbola is a mean proportional between the axes of the ellipse.

27. The tangent at a point P of a rectangular hyperbola meets a diameter QCQ' in T. Shew that CQ and TQ' subtend equal angles at P.



Draw through P the circular section QP, cutting APA' in PN, so that PN is at right angles to AA' and therefore parallel to XK.

Let the generating line through P meet the circle EF in R; and join SP.

Then PS and PR are tangents to the sphere;

$$\therefore SP = PR = EQ.$$

But $$EQ : NX :: AE : AX$$
$$:: SA : AX,$$

and $$NX = PK,$$

$$\therefore SP : PK :: SA : AX.$$

Also, AE being less than AX, SA is less than AX, and the curve APA' is therefore an ellipse, of which S is the focus and XK the directrix.

If another sphere be inscribed in the cylinder touching AA' in S', S' is the other focus, and the corresponding directrix is the intersection of the plane of contact $E'F'$ with APA'.

Producing the generating line RP to meet the circle $E'F'$ in R' we observe that $S'P=PR'$, and therefore

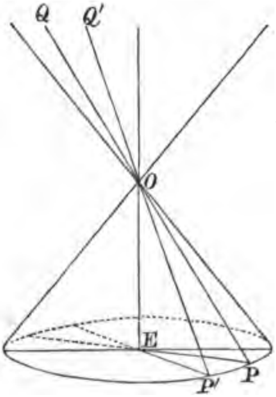
$$\begin{aligned} SP+S'P &= RR' = EE' \\ &= AE+AE'. \\ &= AS+AS' \\ &= AA'. \end{aligned}$$

Since $\qquad AS'=AE'=A'F=A'S.$

The transverse axis of the section is AA' and the conjugate, or minor axis is evidently a diameter of a circular section.

143. Def. If O be a fixed point in a straight line OE drawn through the centre E of a fixed circle at right angles to the plane of the circle, and if a straight line QOP move so as always to pass through the circumference of the circle, the surface generated by the line QOP is called a *Right Circular Cone.*

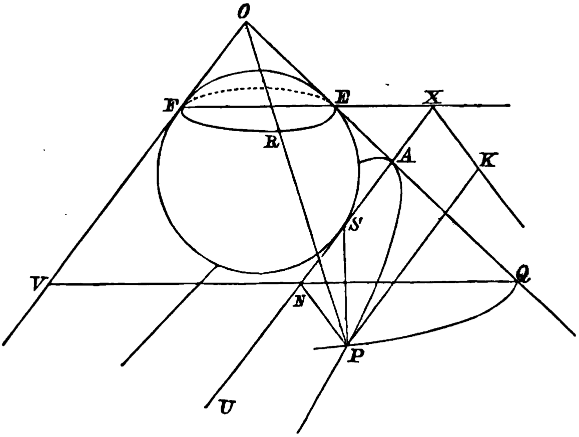
The line OE is called the *axis* of the cone, the point O is the *vertex*, and the constant angle POE is the semi-vertical angle of the cone.



It is evident that any section by a plane perpendicular to the axis, or parallel to the base of the cone, is a circle; and that any section by a plane through the vertex consists of two straight lines, the angle between which is greatest and equal to the vertical angle when the plane contains the axis.

Any plane containing the axis is called a *Principal Section.*

144. Prop. II. *The section of a cone by a plane, which is not perpendicular to the axis, and does not pass through the vertex, is either an Ellipse, a Parabola, or an Hyperbola.*



Let *UAP* be the cutting plane, and let the plane of the paper be that principal section which is perpendicular to the plane *UAP*, *OV*, *OAQ* being the generating lines in the plane of the paper.

Let *AU* be the intersection of the principal section *VOQ* by the plane *PAU* perpendicular to it, and cutting the cone in the curve *AP*.

Inscribe a sphere in the cone, touching the cone in the circle EF and the plane AP in the point S, and let XK be the intersection of the planes AP, EF. Then XK is perpendicular to the plane of the paper.

Taking any point P in the curve, join OP cutting the circle EF in R, and join SP.

Draw through P the circular section QPV cutting the plane AP in PN which is therefore perpendicular to AN and parallel to XK.

Then, SP and PR being tangents to the sphere,

$$SP = PR = EQ\,;$$

and
$$EQ : NX :: AE : AX$$
$$:: AS : AX.$$

Also
$$NX = PK;$$

$$\therefore SP : PK :: SA : AX.$$

The curve AP is therefore an Ellipse, Parabola, or Hyperbola, according as SA is less than, equal to, or greater than AX. In any case the point S is a focus and the corresponding directrix is the intersection of the curve with the plane of contact of the sphere.

(1) If AU be parallel to OV, the angle $AXE = OFE = OEF = AEX$, and therefore

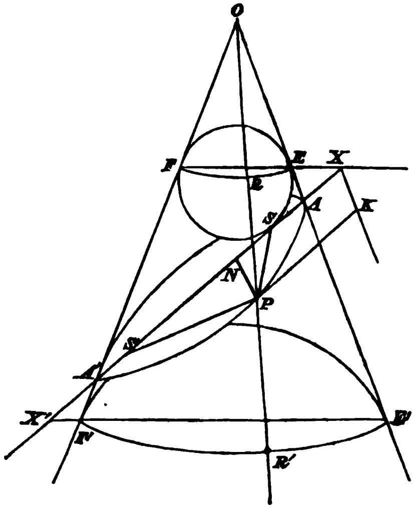
$$SA = AE = AX,$$

and the section is therefore a parabola when the cutting plane is parallel to a generating line, and perpendicular to the principal section which contains the generating line.

(2) Let the line AU meet the curve again in the point A' on the same side of the vertex as the point A.

Then the angle

$$AEX = OFE$$
$$> FXA,$$



and therefore $AE < AX$,

that is $SA < AX$,

and the curve is an ellipse.

In this case another sphere can be inscribed in the cone, touching the cone along the circle $E'F'$ and touching the plane AP in S'.

It may be shewn as before that S' is a focus and that the corresponding directrix is the intersection of the planes $E'F'$, APA'.

(3) Let the line UA produced meet the cone on the other side of the vertex. The section then consists of two separate branches.

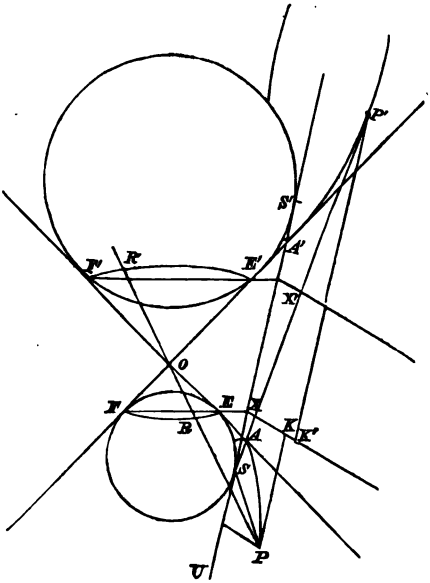
Also the angle $AEX = A'FX$

$< AXF$,

and therefore $AE > AX$,

that is $AS > AX$,

and the curve AP is one branch of an hyperbola, the other branch being the section $A'P'$.



Taking P' in the other branch the proof is the same as before that

$$SP' : P'K' :: SA : AX.$$

In this case a sphere can be inscribed in the other branch of the cone, touching the cone along the circle $E'F'$, and the plane $UA'P'$ in S', and it can be shewn that S' is the other focus of the hyperbola, and that the directrix is the intersection of the cutting plane with the plane of contact $E'F'$.

Hence the section of a cone by a plane cutting in AU the principal section VOQ perpendicular to it is an Ellipse, Parabola, or Hyperbola, according as the angle EAX is greater than, equal to, or less than, the vertical angle of the cone.

Further it is obvious that, if any plane be drawn parallel to the plane AP, the ratio of AE to AX is always the same; hence it follows that all parallel sections have the same eccentricity.

145. This method of determining the focus and directrix was published by Mr Pierce Morton, of Trinity College, in the first Volume of the Cambridge *Philosophical Transactions.*

The method was very nearly obtained by Hamilton, who gave the following construction.

First finding the vertex and focus, A and S, take AE along the generating line equal to AS, and draw the circular section through E; the directrix will be the line of intersection of the plane of the circle with the given plane of section.

Hamilton also demonstrated the equality of SP and PR.

146. PROP. III. *To prove that, in the case of an elliptic section,*

$$SP+S'P=AA'.$$

Taking the 2nd figure,

$$SP=PR \text{ and } S'P=PR';$$

$$\therefore\ SP+S'P=RR'=EE'$$

$$=AE+AE'$$

$$=AS+AS'.$$

But
$$A'S'=A'F'=FF'-A'F$$

$$=EE'-A'S,$$

also
$$A'S'+SS'=A'S;$$

$$\therefore\ 2A'S'+SS'=EE'.$$

Similarly
$$2AS+SS'=EE';$$

$$\therefore\ A'S'=AS,$$

and
$$AS'=A'S.$$

Hence $$SP+S'P=AA',$$

and the transverse, or major axis $=EE'$.

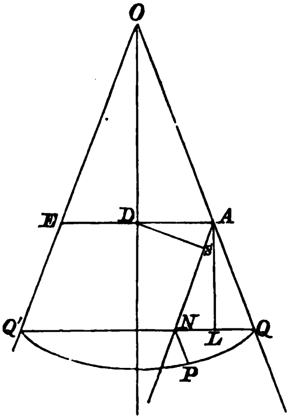
In a similar manner it can be shewn that in an hyperbolic section

$$S'P-SP=AA'.$$

147. **Prop. IV.** *To shew that, in a parabolic section,*

$$PN^2=4AS\,.\,AN.$$

Let A be the vertex of the section, and let ADE be the diameter of the circular section through A.



From D let fall DS perpendicular to AV.

Then $$PN^2=QN\,.\,NQ'$$
$$=QN\,.\,AE$$
$$=4NL\,.\,AD$$

if AL be perpendicular to NQ.

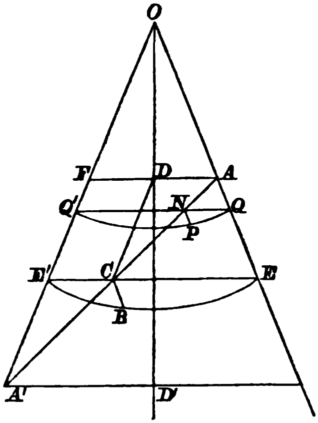
But the triangles ANL, ADS being similar,

$$NL : AN :: AS : AD;$$
$$\therefore NL\,.\,AD=AN\,.\,AS,$$

and $$PN^2=4AS\,.\,AN.$$

148. PROP. V. *To shew that, in an elliptic section,* PN^2 *is to* $AN \cdot NA'$ *in a constant ratio.*

Draw through P the circular section QPQ'.



Then, by a property of the circle,

$$PN^2 = QN \cdot NQ'.$$

Bisect AA' in C, and draw through C the circular section EBE'.

Then $\qquad QN : AN :: CE : AC,$

and $\qquad NQ' : NA' :: CE' : AC;$

$$\therefore\ QN \cdot NQ' : AN \cdot NA' :: EC \cdot CE' : AC^2,$$

or $\qquad PN^2 : AN \cdot NA' :: EC \cdot CE' : AC^2.$

The transverse axis being AA', the square of the semi-minor axis $= BC^2 = EC \cdot CE'$.

But, if ADF be perpendicular to the axis, $AD = DF$, and, AC being equal to CA',

$$CD \text{ is parallel to } A'F,$$

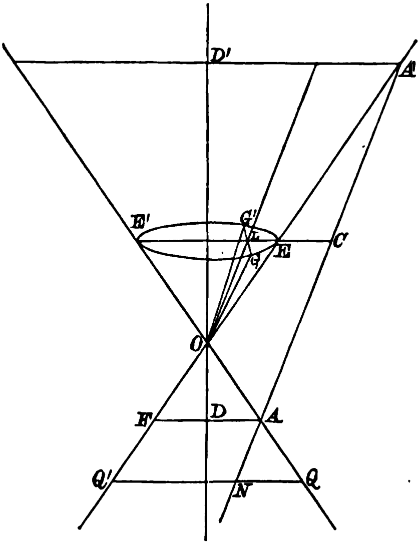
and therefore $\qquad CE' = FD = AD.$

Similarly, $CE = A'D'$, the perpendicular from A' on the axis;

$$\therefore BC^2 = AD \,.\, A'D',$$

that is, *the semi-minor axis is a mean proportional between the perpendiculars from the vertices on the axis of the cone.*

In exactly the same manner it can be shewn for an hyperbolic section, that



$$PN^2 : AN \,.\, NA' :: CE \,.\, CE' : AC^2,$$

and that $$CE = AD,$$

and $$CE' = A'D'.$$

Hence $BC^2 = AD \,.\, A'D'$, AA' being the transverse axis.

We observe also that the conjugate axis is equal to the tangent from C to the circular section passing through C.

149. Prop. VI. *The two straight lines in which a cone is intersected by a plane through the vertex parallel to an hyperbolic section are parallel to the asymptotes of the section.*

Taking the preceding figure, let the parallel plane cut the cone in the lines OG, OG', and the circular section through C in the line GLG', which will be perpendicular to the plane of the paper, and therefore perpendicular to EE' and to OL.

Hence $$GL^2 = EL \,.\, E'L.$$

But $$EL : EC :: OL : A'C,$$

and $$E'L : E'C :: OL : AC;$$

$$\therefore GL^2 : EC \,.\, E'C :: OL^2 : AC^2,$$

or $$GL : OL :: BC : AC;$$

therefore, Art. (98), OG and OG' are parallel to the asymptotes of the hyperbola.

Hence for all parallel hyperbolic sections, the asymptotes are parallel to each other.

If the hyperbola be rectangular, the angle GOG' is a right angle; but this is evidently not possible if the vertical angle of the cone be less than a right angle.

When the vertical angle of the cone is not less than a right angle, and when GOG' is a right angle, LOG is half a right angle, and therefore

$$OL = LG,$$

and $$2 \,.\, OL^2 = OG^2 = OE^2,$$

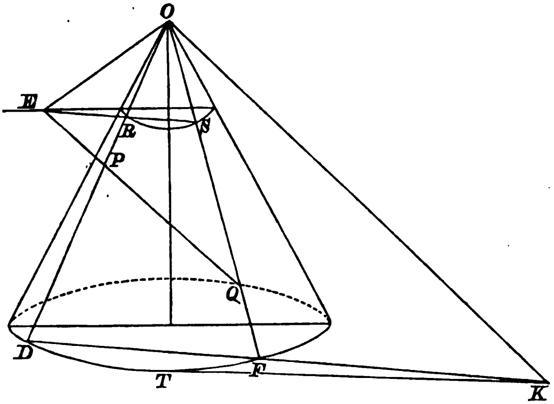
and the length OL is easily constructed.

Hence, placing OL, and drawing the plane GOG' perpendicular to the principal section through OL, any section by a plane parallel to GOG' is a rectangular hyperbola.

150. Prop. VII. *If two straight lines be drawn through any point, parallel to two fixed lines, and intersecting a given cone, the ratio of the rectangles formed by*

the segments of the lines will be independent of the position of the point.

Thus, if through E, the lines EPQ, $EP'Q'$ be drawn, parallel to two given lines, and cutting the cone in the



points P, Q and P', Q', the ratio of $EP.EQ$ to $EP'.EQ'$ is constant.

Through O draw OK parallel to the given line to which EPQ is parallel, and let the plane through OK, EPQ, which contains the generating lines OP, OQ, meet the circular section through E in R and S, and the plane base in the straight line DFK, cutting the circular base in D and F.

Then DFK and ERS being sections of parallel planes by a plane are parallel to each other.

Also, EPQ is parallel to OK;

Therefore ERP, ODK are similar triangles, as are also ESQ, OFK;

$$\therefore EP : ER :: OK : DK,$$

and $$EQ : ES :: OK : FK;$$

$$\therefore EP.EQ : ER.ES :: OK^2 : DK.FK$$
$$:: OK^2 : KT^2,$$

if KT be the tangent to the circular base from K.

If a similar construction be made for $EP'Q'$ we shall have

$$EP'.EQ' : ER'.ES' :: OK'^2 : K'T'^2.$$

But $$ER.ES=ER'.ES';$$

therefore the rectangles $EP.EQ$ and $EP'.EQ'$ are each in a constant ratio to the same rectangle, and are therefore in a constant ratio to each other.

Since the plane through EPQ, $EP'Q'$ cuts the cone in an ellipse, parabola, or hyperbola, this theorem includes as particular cases those of Arts. (49), (76), (86), (119) and (129).

The proof is the same if the point P be within the cone, or if one or both of the lines meet opposite branches of the cone.

If the chords be drawn through the centre of the section PEP', the rectangles become the squares of the semidiameters.

Hence the parallel diameters of all parallel sections of a cone are proportional to each other.

If the lines move until they become tangents the rectangles then become the squares of the tangents; therefore if a series of points be so taken that the tangents from them are parallel to given lines, these tangents are always in the same proportion. The locus of the point E will be the line of intersection of two fixed planes touching the cone, that is, a fixed line through the vertex.

EXAMPLES.

1. SHEW how to cut from a cylinder an ellipse whose eccentricity shall be the same as the ratio of the side of a square to its diagonal.

2. Shew how to cut from a cone an ellipse whose eccentricity is the ratio of one to two.

3. Find the least angle of a cone from which it is possible to cut an hyperbola, whose eccentricity shall be the ratio of two to one.

4. Shew that all sections of a right cone, made by planes parallel to a tangent plane of the cone, are parabolas, and that the foci lie on a cone having with the first a common vertex and axis.

5. The centre of a spherical ball is moveable in a vertical plane which is equidistant from two candles of the same height on a table; find its locus when the two shadows on the ceiling are always in contact.

6. Prove that all sections of a cone by parallel planes are conics having the same eccentricity.

7. Find the locus of the foci of the sections made by a series of parallel planes.

8. Give a geometrical construction, by which a cone may be cut, so that the section may be an ellipse of given eccentricity.

9. If two plane sections of a right cone be taken, having the same directrix, the foci corresponding to that directrix lie on a straight line which passes through the vertex.

10. Different elliptic sections of a right cone are taken, having equal major axes; shew that the locus of the centres of the sections is a spheroid, and determine in what cases it is oblate or prolate respectively.

11. The vertex of a cone and the centre of a sphere inscribed within it are given in position: a plane section of the cone, at right angles to any generating line of the cone, touches the sphere: prove that the locus of the point of contact is a surface generated by the revolution of a circle, which touches the axis of the cone at the centre of the sphere.

12. Given a right cone and a point within it, there are but two sections which have this point for focus; and the planes of these sections make equal angles with the straight line joining the given point and the vertex of the cone.

13. If the curve formed by the intersection of any plane with a cone be projected upon a plane perpendicular to the axis; prove that the curve of projection will be a conic section having its focus at the point in which the axis meets the plane of projection.

14. An ellipse and an hyperbola are so situated that the vertices of each curve are the foci of the other, and the curves are in planes at right angles to each other. If P be a point on the ellipse, and Q a point on the hyperbola, S the vertex, and A the interior focus of that branch of the hyperbola, then

$$PQ + AS = PS + AQ.$$

15. The shadow of a ball is cast by a candle on an inclined plane in contact with the ball; prove that, as the candle burns down, the locus of the centre of the shadow will be a straight line.

16. If sections of a right cone be made, perpendicular to a given plane, such that the distance between a focus of a section and that vertex which lies on the same generating line parallel to the given plane be constant, prove that the transverse axes, produced if necessary, of all sections will touch one of two fixed circles.

17. If the vertical angle of a cone, vertex V, be a right angle, P any point of a parabolic section, and PN perpendicular to the axis of the parabola,

$$VP = 2\,AS + AN,$$

A being the vertex and S the focus.

18. If two cones be described touching the same two spheres, the eccentricities of the two sections of them made by the same plane bear to one another a ratio constant for all positions of the plane.

19. Given a right cone and a point within it, there are but two sections which have this point for focus; and the planes of these sections make equal angles with the straight line joining the given point and the vertex of the cone.

20. The vertex of any right circular cone which contains a given ellipse will lie on a certain hyperbola, and the axis of the cone will be a tangent to the hyperbola.

21. Different elliptic sections of a right cone are taken such that their minor axes are equal; shew that the locus of their centres is the surface formed by the revolution of an hyperbola about the axis of the cone.

CHAPTER VII.

The Similarity of Conics, the Areas of Conics, and the Curvatures of Conics.

SIMILAR CONICS.

151. DEF. *Conics which have the same eccentricity are said to be similar to each other.*

This definition is justified by the consideration that the character of the conic depends on its eccentricity alone, while the dimensions of all parts of the conic are entirely determined by the distance of the focus from the directrix.

Hence, according to this definition, all parabolas are similar curves.

152. PROP. I. *If radii be drawn from the vertices of two parabolas making equal angles with the axes, these radii are always in the same proportion.*

Let AP, ap be the radii, PN and pn the ordinates, the angles PAN, pan, being equal.

Then $AN : an :: AP : ap$,

and $PN : pn :: AP : ap$.

But $PN^2 : pn^2 :: AS.AN : as.an$;

$\therefore AP^2 : ap^2 :: AS.AP : as.ap$,

or $AP : ap :: AS : as$.

It can also be shewn that focal radii making equal angles with the axes are always in the same proportion.

153. PROP. II. *If two ellipses be similar their axes are in the same proportion, and any other diameters, making equal angles with the respective axes, are in the proportion of the axes.*

Let CA, CB be the semi-axes of one ellipse, ca, cb of the other, and CP, cp two radii such that the angle $PCA = pca$.

Then, since the eccentricities are the same, we have, if S, s be foci,

$$AC : SC :: ac : sc;$$

$$\therefore\ AC^2 : AC^2 - SC^2 :: ac^2 : ac^2 - sc^2,$$

or
$$AC^2 : BC^2 :: ac^2 : bc^2.$$

Hence it follows, if PN, pn be ordinates, that

$$PN^2 : AC^2 - CN^2 :: pn^2 : ac^2 - cn^2;$$

but, by similar triangles,

$$PN : pn :: CN : cn,$$

therefore $CN^2 : AC^2 - CN^2 :: cn^2 : ac^2 - cn^2;$

and
$$CN^2 : AC^2 :: cn^2 : ac^2.$$

Hence
$$CP : cp :: CN : cn$$
$$:: AC : ac.$$

So also lines drawn similarly from the foci, or any other corresponding points of the two figures will be in the ratio of the transverse axes.

Exactly the same demonstration is applicable to the hyperbola, but in this case, if the ratio of SC to AC in two hyperbolas be the same, it follows from Art. (98) that the angle between the asymptotes is the same in both curves.

In the case of hyperbolas we have thus a very simple test of similarity.

The Areas bounded by Conics.

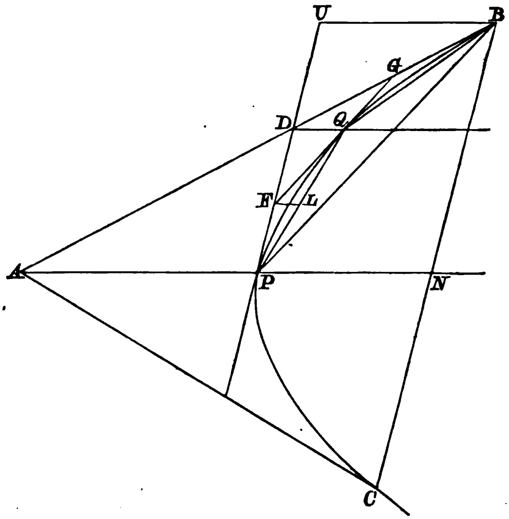
154. Prop. III. *If AB, AC be two tangents to a parabola, the area between the curve and the chord BC is two-thirds of the triangle ABC.*

Draw the tangent DPE parallel to BC; then

$$AP = PN,$$

and $$BC = 2 . DE;$$

therefore triangle $$BPC = 2ADE.$$



Again, draw the diameter DQM meeting BP in M.

By the same reasoning, FQG being the tangent parallel to BP, the triangle $PQB = 2FDG$.

Through F draw the diameter FRL, meeting PQ in L, and let this process be continued indefinitely.

Then the sum of the triangles within the parabola is double the sum of the triangles without it.

But, since the triangle BPC is half ABC, it is greater than half the parabolic area $BQPC$;

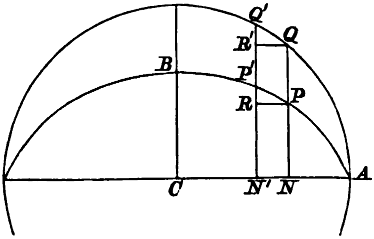
Therefore, Euclid, Bk. XII., the difference between the parabolic area and the sum of the triangles can be made ultimately less than any assignable quantity;

And, the same being true of the outer triangles, it follows that the area between the curve and BC is double of the area between the curve and AB, AC, and is therefore two-thirds of the triangle ABC.

Cor. Since PN bisects every chord parallel to BC it bisects the parabolic area BPC; therefore, completing the parallelogram $PNBU$, the parabolic area BPN is two-thirds of the parallelogram UN.

155. Prop. IV. *The area of an ellipse is to the area of the auxiliary circle in the ratio of the conjugate to the transverse axis.*

Draw a series of ordinates, QPN, $Q'P'N'$, ... near each other, and draw PR, QR' parallel to AC.



Then, since

$$PN : QN :: BC : AC,$$

the area

$$PN' : QN' :: BC : AC,$$

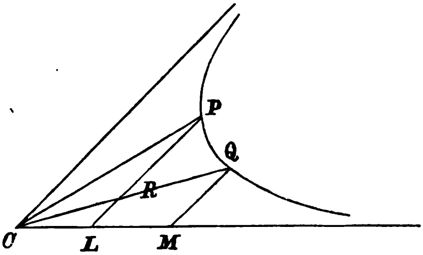
and, this being true for all such areas, the sum of the parallelograms PN is to the sum of the parallelograms QN as BC to AC.

But, if the number be increased indefinitely, the sums of these parallelograms ultimately approximate to the areas of the ellipse and circle.

Hence the ellipse is to the circle in the ratio of BC to AC.

The student will find in Newton's 2nd and 3rd Lemmas (*Principia*, Section i.), a formal proof of what we have here assumed as sufficiently obvious, that the sum of the parallelograms PN is ultimately equal to the area of the ellipse.

156. Prop. V. *If P, Q be two points of an hyperbola, and if PL, QM parallel to one asymptote meet the other in L and M, the hyperbolic sector CPQ is equal to the hyperbolic trapezium $PLMQ$.*



For the triangles CPL, CQM are equal, and, if PL meet CQ in R, it follows that the triangle CPR = the trapezium $LRQM$; hence, adding to each the area RPQ the theorem is proved.

157. Prop. VI. *If points L, M, N, K be taken in an asymptote of an hyperbola, such that*

$$CL : CM :: CN : CK,$$

and if LP, MQ, NR, KS, parallel to the asymptote, meet the curve in P, Q, R, S, the hyperbolic areas CPQ, CRS will be equal.

Let QR and PS produced meet the asymptotes in F, F', G, G';

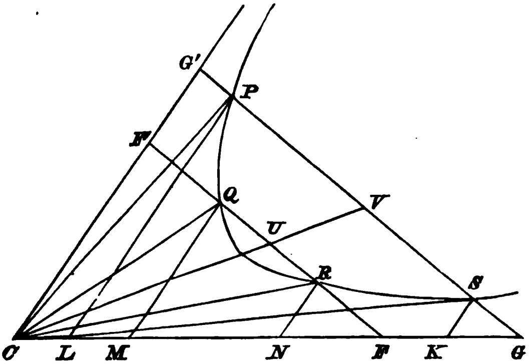
then $\quad RF = QF'$ and $SG = PG'$, Art. (116);

$$\therefore NF = CM \text{ and } KG = CL.$$

Hence
$$\begin{aligned} NF : KG &:: CM : CL \\ &:: CK : CN \\ &:: RN : SK, \end{aligned}$$

and therefore SP is parallel to QR.

The diameter CUV conjugate to PS bisects all chords parallel to PS, and therefore bisects the area $PQRS$;



also the triangle $\quad CPV = CSV$,

and $\quad CQU = CUR$;

therefore taking from CPV and CSV the equal triangles CQU, CRU, and the equal areas $PQUV$, $SRUV$, the remaining areas, which are the hyperbolic sectors CPQ, CRS, are equal.

Cor. Hence if a series of points, L, M, N,... be taken such that CL, CM, CN, CK,... are in continued proportion, it follows that the hyperbolic sectors CPQ, CQR, CRS, &c. will be all equal.

It will be noticed in this case that the tangent at Q will be parallel to PR, the tangent at R parallel to QS, and so also for the rest.

The Curvature of Conics.

158. Def. If a circle touch a conic at a point P, and pass through another point Q of the conic, and if the point Q move near to, and ultimately coincide with P, the circle, in its ultimate condition, is called the circle of curvature at P.

Prop. VII. *The chord of intersection of a conic with the circle of curvature at any point is inclined to the axis at the same angle as the tangent at the point.*

It has been shewn that, if a circle intersect a conic in four points P, Q, R, V, the chords PQ, RV are equally inclined to the axis.

Let P and Q coincide with each other; then the tangent at P and the chord RV are equally inclined to the axis.

Let the point V now approach to and coincide with P; the circle becomes the circle of curvature at P, and the chord VR becomes PR the chord of intersection.

Hence PR and the tangent at P are equally inclined to the axis.

159. **Prop. VIII.** *If the tangent at any point P of a parabola meet the axis in T, and if the circle of curvature at P meet the curve in Q,*

$$PQ = 4 . PT.$$

Draw the ordinate PNP'; then taking the figure of the next article, TP' is the tangent at P', and the angle $P'TF = PTF = PFT$; therefore PQ is parallel to TP', and is bisected by the diameter $P'E$.

Hence $$PQ = 2 . PE = 4 P'T$$
$$= 4 PT.$$

160. **Prop. IX.** *To find the chord of curvature through the focus and the diameter of curvature at any point of a parabola.*

Let the circle meet PS produced in V, and the normal PG, produced, in O.

The angle $$PFS = PTS = SPT$$
$$= PQV,$$

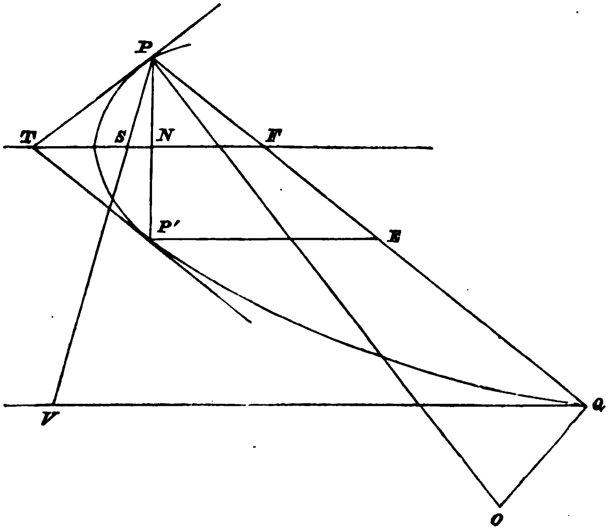
since PT is a tangent.

Therefore QV is parallel to the axis,

and $$PV : SP :: PQ : PF.$$

Hence $$PV = 4 . SP.$$

Again, the angle $POQ = PVQ = PSN$;



$$\therefore PO : PQ :: SP : PN,$$

or $$PO : SP :: 4PT : PN$$
$$:: 4SP : SY,$$

if SY be perpendicular to PT.

Cor. 1. Since the normal bisects the angle between SP and the diameter through P, it follows that the chord of curvature parallel to the axis is $4SP$.

Cor. 2. The diameter of curvature, PO, may also be expressed as follows:

Let GL be the perpendicular from G on SP; then PL = the semi-latus rectum $= 2AS$.

Also PVO being a right angle,

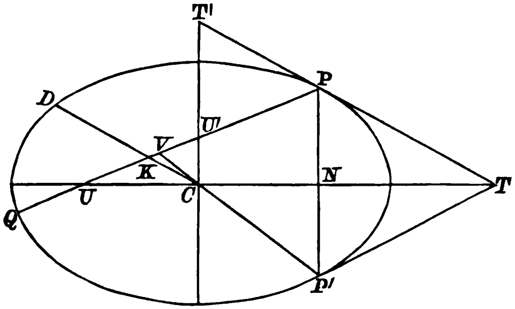
$$PO : PG :: PV : PL$$
$$:: 4SP : PL$$
$$:: 4SP . PL : PL^2,$$

but $$4SP\,.\,PL = 8SP\,.\,AS = 8SY^2 = 2PG^2;$$

$$\therefore PO : PG :: 2PG^2 : PL^2.$$

161. PROP. X. *If the chord of intersection, PQ, of an ellipse, or hyperbola, with the circle of curvature at P, meet CD, the semi-diameter conjugate to CP, in K,*

$$PQ\,.\,PK = 2CD^2.$$



Drawing the ordinate PNP', the tangent at P' is parallel to PQ, as in the parabola, and PQ is therefore bisected in V, by the diameter CP'.

Let PQ meet the axes in U and U'; then, $U'C$ being parallel to PP',

$$PV : PU' :: VP' : CP'$$

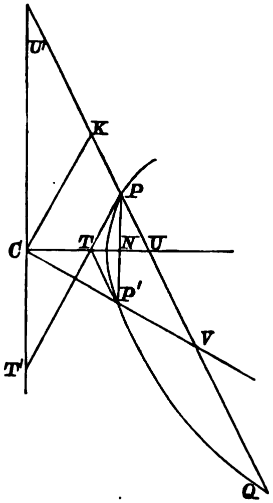
$$:: UT : CT,$$

since PU, $P'T$ are parallel.

Also $$UT : CT :: PU : PK;$$

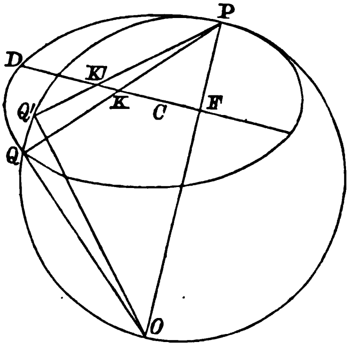
$$\therefore PV.PK = PU.PU' = PT.PT' = CD^2,$$

and $$PQ\,.\,PK = 2\,.\,CD^2.$$



162. Prop. XI. *If the chord of curvature PQ', of an ellipse or hyperbola in any direction, meet CD in K',*

$$PQ' . PK' = 2 . CD^2.$$



Let PO be the diameter of curvature meeting CD in F; then PQO, $PQ'O$ are right angles, and a circle can be drawn through $Q'K'FO$;

$$\therefore PQ' . PK' = PF . PO$$
$$= PK . PQ = 2 . CD^2.$$

Cor. 1. Hence PO being the diameter of curvature,

$$PF . PO = 2 . CD^2.$$

Cor. 2. If PQ' pass through the focus,

$$PK' = AC,$$

and $$PQ' . AC = 2 . CD^2.$$

Cor. 3. If PQ' pass through the centre,

$$PQ' . CP = 2 . CD^2.$$

163. We can also express the diameter of curvature as follows:

PG being the normal, let GL be perpendicular to SP, and let PR be the chord of curvature through S.

Then GL is parallel to OR,

and $$PO : PG :: PR : PL$$
$$:: PR . PL : PL^2.$$

But $$PR . AC = 2 . CD^2;$$

$$\therefore PR : AC :: 2 . CD^2 : AC^2$$
$$:: 2 . PG^2 : BC^2,$$

and $$PR . PL : AC . PL :: 2 . PG^2 : BC^2.$$

But, PL being equal to the semi-latus rectum,

$$PL . AC = BC^2;$$

$$\therefore PR . PL = 2 . PG^2,$$

and $$PO : PG :: 2PG^2 : PL^2.$$

Hence, in any conic, the radius of curvature at any point is to the normal at the point as the square of the normal to the square of the semi-latus rectum.

EXAMPLES.

1. The radius of curvature at the extremity of the latus rectum of a parabola is equal to twice the normal.

2. The circle of curvature at the end of the latus rectum intersects the parabola on the normal at that point.

3. The chord of curvature at a point P in the parabola passing through the vertex A is to $4PY :: PY : AP$.

4. The circle of curvature at a point P in a parabola, cuts off from the diameter at P a portion equal to the parameter of that diameter.

5. P and p are points on a parabola on the same side of the axis; PN and pn are perpendiculars on the axis; the normals P and p meet at a point Q: shew that the distance of Q from the axis is

$$\frac{2\,PN\,.\,pn\,(PN+pn)}{(\text{latus rectum})^2}.$$

Deduce the value of the radius of curvature at any point of a parabola.

6. If P be a point of an ellipse equidistant from the axis minor and one of the directrices, prove that the circle of curvature at P will pass through one of the foci.

7. The chord of curvature through the focus, at any point, is equal to the focal chord parallel to the tangent at the point.

8. Prove that the locus of the middle points of the common chords of a given parabola and its circles of curvature is a parabola, and that the envelope of the chords is also a parabola.

9. The circles of curvature at the extremities P, D of two conjugate diameters of an ellipse meet the ellipse again in Q, R, respectively, shew that PR is parallel to DQ.

10. In the rectangular hyperbola, the radius of curvature at a point P varies as CP^3.

11. The tangent at a point P of an ellipse whose centre is C meets the axis in T and t; if CP produced meet in L the circle described about the triangle TCt, shew that PL is half the chord of curvature at P in the direction of C, and that the rectangle contained by CP, CL, is constant.

12. If P be a point on a conic, Q a point near it, and if QE, perpendicular to PQ, meet the normal at P in E, then ultimately when Q coincides with P, PE is the diameter of curvature at P.

13. Prove that the ultimate point of intersection of consecutive normals is the centre of curvature.

14. If a tangent be drawn from any point of a parabola to the circle of curvature at the vertex, the length of the tangent will be equal to the abscissa of the point measured along the axis.

15. The circle of curvature at a point where the conjugate diameters are equal, meets the ellipse again at the extremity of the diameter.

16. Find the points at which the radius of curvature is a mean proportional between the semi-major and semi-minor axes of an ellipse.

17. The chord of curvature at P perpendicular to the major axis is to PM, the ordinate at P, :: $2 . CD^2 : BC^2$.

18. Prove that there is a point P on an ellipse such that if the normal at P meet the ellipse in Q, PQ is a chord of the circle of curvature at P, and find its position.

19. If SP be the focal distance of a point P of a parabola, and SQ, perpendicular to SP, meet the normal at P in Q, PQ is half the radius of curvature at P.

20. The chord of curvature at a point P of a rectangular hyperbola, perpendicular to an asymptote, is to CD :: $CD : 2 . PN$, where PN is the distance of P from the asymptote.

21. If G be the foot of the normal at a point P of an ellipse, and GK perpendicular to PG, meet CP in K, then KE, parallel to the axis minor, will meet PG in the centre of curvature at P.

22. In the rectangular hyperbola, if CP be produced to Q, so that $PQ = CP$, and QO be drawn perpendicular to CQ to intersect the normal in O, O is the centre of curvature at P.

23. If S and H be the foci of an ellipse, B the extremity of the axis minor, a circle described through S, H, and B, will cut the minor axis in the centre of curvature at B.

24. The tangent at any point P in an ellipse, of which S and H are the foci, meets the axis major in T, and TQR bisects HP in Q and meets SP in R; prove that PR is one-fourth of the chord of curvature at P through S.

25. An ellipse, a parabola, and an hyperbola, have the same vertex and the same focus; shew that the curvature, at the vertex, of the parabola is greater than that of the hyperbola, and less than that of the ellipse.

CHAPTER VIII.

PROJECTIONS.

164. **Def.** The projection of a point on a plane is the foot of the perpendicular let fall from the point on the plane.

If from all points of a given curve perpendiculars be let fall on a plane, the curve formed by the feet of the perpendiculars is the projection of the given curve.

The projection of a straight line is also a straight line, for it is the line of intersection with the given plane of a plane through the line perpendicular to the given plane.

Parallel straight lines project into parallel lines, for the projections are the lines of intersection of parallel planes with the given plane.

165. **Prop. I.** *Parallel straight lines, of finite lengths, are projected in the same ratio.*

That is, if ab, pq be the projections of the parallel lines AB, PQ,

$$ab : AB :: pq : PQ.$$

For, drawing AC parallel to ab and meeting Bb in C, and PR parallel to pq and meeting Qq in R, ABC and PQR are similar triangles; therefore

$$AC : AB :: PR : PQ,$$

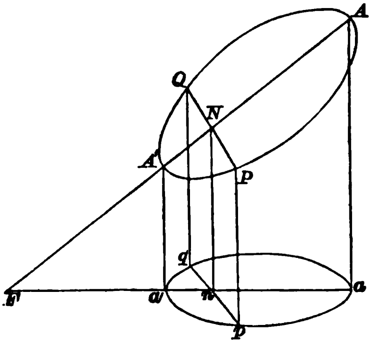
and $$AC = ab, \quad PR = pq.$$

166. **Prop. II.** *The projection of the tangent to a curve at any point is the tangent to the projection of the curve at the projection of the point.*

For if p, q be the projections of two points P, Q of a curve, the line pq is the projection of the line PQ, and when the line PQ turns round P until Q coincides with P, pq turns round p until q coincides with p, and the ultimate position of pq is the tangent at p.

167. Prop. III. *The projection of a circle is an ellipse.*

Let aba' be the projection of the circle ABA'.



Take a chord PQ parallel to the plane of projection, then its projection $pq = PQ$.

Let the diameter ANA' perpendicular to PQ meet in F the plane of projection, and let $aa'F$ be the projection of $AA'F$.

Then aa' bisects pq at right angles in the point n, and

$$an : AN :: aF : AF,$$

$$a'n : A'N :: AF : AF;$$

$$\therefore AN . NA' : an . na' :: AF^2 : aF^2;$$

but
$$AN . NA' = PN^2 = pn^2,$$

$$\therefore pn^2 : an . na' :: AF^2 : aF^2,$$

and the curve apa' is an ellipse, having its axes in the ratio of

$$aF : AF, \text{ or of } aa' : AA'.$$

Moreover, since we can place the circle so as to make the ratio of aa' to AA' whatever we please, an ellipse of any eccentricity can be obtained.

In this demonstration we have assumed only the property of the principal diameters of an ellipse. Properties of other diameters can be obtained by help of the preceding theorems, as in the following instances.

168. PROP. IV. *The locus of the middle points of parallel chords of an ellipse is a straight line.*

For, projecting a circle, the parallel chords of the ellipse are the projections of parallel chords of the circle, and as the middle points of these latter lie in a diameter of the circle, the middle points of the chords of the ellipse lie in the projection of the diameter which is a straight line, and is a diameter of the ellipse.

Moreover, the diameter of the circle is perpendicular to the chords it bisects; hence

Perpendicular diameters of a circle project into conjugate diameters of an ellipse.

169. PROP. V. *If two intersecting chords of an ellipse be parallel to fixed lines, the ratio of the rectangles contained by their segments is constant.*

Let OPQ, ORS be two chords of a circle, parallel to fixed lines, opq, ors their projections.

Then $OP.OQ$ is to $op.oq$ in a constant ratio, and $OR.OS$ is to $or.os$ in a constant ratio; but

$$OP.OQ = OR.OS.$$

Therefore $op.oq$ is to $or.os$ in a constant ratio; and opq, ors are parallel to fixed lines.

170. PROP. VI. *If qvq' be a double ordinate of a diameter cp, and if the tangent at q meet cp produced in t,*

$$cv.ct = cp^2.$$

The lines qvq' and cp are the projections of a chord QVQ' of a circle which is bisected by a diameter CP, and t is the projection of T the point in which the tangent at Q meets CP produced.

But, in the circle,

$$CV \cdot CT = CP^2,$$

or

$$CV : CP :: CP : CT;$$

and, these lines being projected in the same ratio, it follows that

$$cv : cp :: cp : ct,$$

or

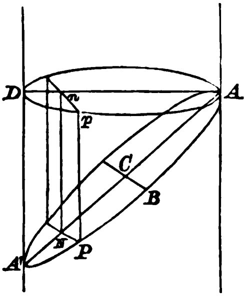
$$cv \cdot ct = cp^2.$$

Hence it follows that tangents to an ellipse at the ends of any chord meet in the diameter conjugate to the chord.

The preceding will serve as sufficient illustrations of the application of the method.

171. **Prop. VII.** *An ellipse can be projected into a circle.*

This is really the converse of Art. (166), but we give a construction for the purpose.



Draw a plane through AA', the transverse axis, perpendicular to the plane of the ellipse, and in this plane describe a circle on AA' as diameter. Also take the chord AD, equal to the conjugate axis, and join $A'D$, which is perpendicular to AD.

Through AD draw a plane perpendicular to $A'D$, and project a principal chord PNP' on this plane.

Then $PN^2 : AN.NA' :: BC^2 : AC^2$.

But $PN = pn$,

$$An : AN :: AD : AA'$$
$$:: BC : AC,$$

and $Dn : A'N :: BC : AC$.

Hence $An.nD : AN.NA' :: BC^2 : AC^2$,

and therefore $pn^2 = An.nD$,

and the projection ApD is a circle.

This theorem, in the same manner as that of Art. (166), may be employed in deducing properties of oblique diameters and oblique chords of an ellipse.

172. PROP. VII. *The projection of a parabola is a parabola.*

For if PNP' be a principal chord, bisected by the axis AN, the projection pnp' will be bisected by the projection an.

Moreover $pn : PN$ will be a constant ratio, as also will be $an : AN$.

And $PN^2 = 4AS.AN$.

Hence pn^2 will be to $4AS.an$ in a constant ratio, and the projection is a parabola, the tangent at a being parallel to pn.

173. PROP. VIII. *An hyperbola can be always projected into a rectangular hyperbola.*

For the asymptotes can be projected into two straight lines cl, cl' at right angles, and if PM, PN be parallels to the asymptotes from a point P of the curve, $PM.PN$ is constant.

But $pm : PM$ and $pn : PN$ are constant ratios;

$\therefore$ $pm.pn$ is constant.

And since pm and pn are perpendicular respectively to cl and cl', it follows that the projection is a rectangular hyperbola.

The same proof evidently shews that any projection of an hyperbola is also an hyperbola.

The explanations of this chapter apply to a particular case only of the general method of projections; in strictness we should have defined the method as the method of orthogonal projections.

EXAMPLES.

1. Prove by the method of projections the theorems of Arts. 65, 70, 71, 74, 75, 76, 77, and 82.

2. A parallelogram is inscribed in a given ellipse; shew that its sides are parallel to conjugate diameters, and find its greatest area.

3. TP, TQ are tangents to an ellipse, and CP', CQ' are parallel semidiameters; PQ is parallel to $P'Q'$.

4. Determine the greatest triangle which can be inscribed in an ellipse having an angular point fixed at a point in its perimeter.

5. If a straight line meet two concentric similar and similarly situated ellipses, the portions intercepted between the curves are equal.

6. Find the locus of the point of intersection of the tangents at the extremities of pairs of conjugate diameters of an ellipse.

7. Find the locus of the middle points of the lines joining the extremities of conjugate diameters.

8. If a tangent be drawn at the extremity of the major axis meeting two equal conjugate diameters CP, CD produced in T and t; then $PD^2 = 2AT^2$.

9. If a chord AQ drawn from the vertex be produced to meet the minor axis in O, and CP be a semidiameter parallel to it, then $AQ \cdot AO = 2CP^2$.

10. If the tangent to an ellipse at P meet a pair of conjugate diameters in T and t, prove that

$$PT \cdot Pt = CD^2.$$

11. If the tangent at P meet a pair of parallel tangents AT, at in T and t, prove that

$$AT \, . \, at = CB^2,$$

$$\text{and} \quad PT \, . \, Pt = CD^2,$$

CB being the semidiameter parallel to AT, and CD conjugate to CP.

12. A parallelogram is inscribed in an ellipse, and from any point on the ellipse two straight lines are drawn parallel to the sides of the parallelogram; prove that the rectangles under the segments of these straight lines, made by the sides of the parallelogram, will be to one another in a constant ratio.

13. A parallelogram circumscribes an ellipse, touching the curve at the extremities of conjugate diameters, and another parallelogram is formed by joining the points where its diagonals meet the ellipse: prove that the area of the inner parallelogram is half that of the outer one.

If four similar and similarly situated ellipses be inscribed in the spaces between the outer parallelogram and the curve, prove that their centres lie in a similar and similarly situated ellipse.

14. If a parallelogram be inscribed in an ellipse, so that the diameter bisecting two opposite sides is always divided by either side in a constant ratio, its area will be constant.

15. If a parallelogram circumscribe an ellipse, and if one of its diagonals bear a constant ratio to the diameter it contains, the area of the parallelogram will be constant.

16. About a given triangle PQR is circumscribed an ellipse, having for centre the point of intersection (C) of the lines from P, Q, R bisecting the opposite sides, and PC, QC, RC are produced to meet the curve in P', Q', R'; shew that, if tangents be drawn at these points, the triangle so formed will be similar to PQR, and four times as great.

17. The locus of the middle points of all chords of an ellipse which pass through a fixed point is an ellipse similar and similarly situated to the given ellipse, and with its centre in the middle point of the line joining the given point and the centre of the given ellipse.

18. Prove that an ellipse can be inscribed in a parallelogram so as to touch the middle points of the four sides, and that it is the greatest of all inscribed ellipses.

19. A polygon of a given number of sides circumscribes an ellipse. Prove that, when its area is a minimum, any side is parallel to the line joining the points of contact of the two adjacent sides.

20. Assuming that the greatest triangle which can be inscribed in a circle is equilateral, prove by the method of projections that the greatest triangle which can be inscribed in an ellipse has one of its sides bisected by a diameter of the ellipse, and the others cut in points of trisection by the conjugate diameter.

21. AB is a given chord of an ellipse, and C any point in the ellipse; shew that the locus of the point of intersection of lines drawn from A, B, C to the middle points of the opposite sides of the triangle ABC is a similar ellipse.

22. CP, CD are conjugate semidiameters of an ellipse; if an ellipse, similar and similarly situated to the given ellipse, be described on PD as diameter, it will pass through the centre of the given ellipse.

23. If an hyperbola and its conjugate are described having a pair of conjugate diameters of an ellipse for asymptotes, and cutting the ellipse in the points a, b, c, d taken in order, shew that the diameters Oa, Oc, and Ob, Od are conjugate diameters in the ellipse; and also that Oa, Od, and Ob, Oc are conjugate diameters in the hyperbolas.

24. PT, pt are tangents at the extremities of any diameter Pp of an ellipse; any other diameter meets PT in T and its conjugate meets pt in t; also any tangent meets PT in T' and pt in t'; shew that $PT : PT' :: pt' : pt$.

25. From the ends P, D of conjugate diameters of an ellipse lines are drawn parallel to any tangent line; from the centre C any line is drawn cutting these lines and the tangent in p, d, t, respectively; prove that

$$Cp^2 + Cd^2 = Ct^2.$$

26. If CP, CD be conjugate diameters of an ellipse, and if BP, BD be joined, and also AD, $A'P$, these latter intersecting in O, the figure $BDOP$ will be a parallelogram.

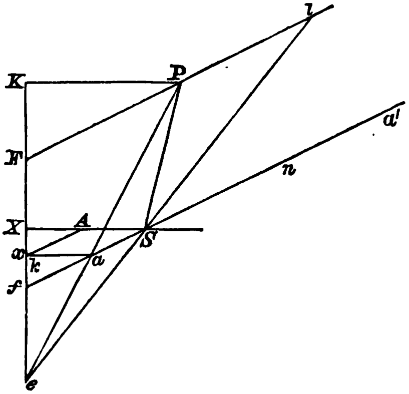
CHAPTER IX.

OF CONICS IN GENERAL.

The Construction of a Conic.

174. The method of construction, given in Chapter I., can be extended in the following manner.

Let fSn be any straight line drawn through the focus S, and draw Ax from the vertex parallel to fS, and meeting the directrix in x.



Divide the line fSn in a and a' so that

$$Sa : af :: Sa' : a'f :: SA : Ax;$$

then a and a' are points on the curve, for if ak be the perpendicular on the directrix,

$$ak : af :: AX : Ax,$$

and therefore $\quad Sa : ak :: SA : AX.$

Take any point e in the directrix, draw the lines eSl, ea through S and a, and draw SP making the angle PSl equal to LSn.

Through P draw FPl parallel to fS, and meeting eS produced in l,

then $$Pl = SP,$$

and $$Pl : PF :: Sa : af;$$

$$\therefore SP : PF :: Sa : af,$$

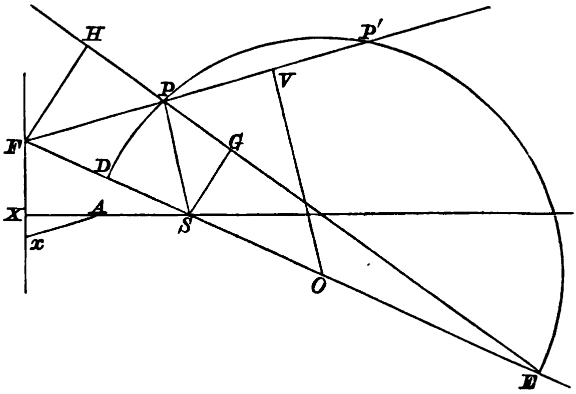
and $$SP : PK :: Sa : ak;$$

therefore P is a point in the curve.

The other point of the curve in the line FP may be found as in Art. 9.

175. The construction for the point (a) gives a simple proof that the tangent at the vertex is perpendicular to the axis. For when the angle ASa is diminished, Sa approaches to equality with SA, and therefore the angle aAS is ultimately a right angle.

176. PROP. I. *To find the points in which a given straight line is intersected by a conic of which the focus, the directrix, and the eccentricity are given.*



Let FPP' be the straight line, and draw Ax parallel to it. Join FS, and find the points D and E such that

$$SD : DF :: SE : EF :: SA : Ax.$$

Describe the circle on DE as diameter, and let it intersect the given line in P and P'.

Join DP, EP and draw SG, FH at right angles to EP.

Then
$$\begin{aligned} SG : FH &:: SE : EF \\ &:: SD : DF \\ &:: PG : PH; \end{aligned}$$

therefore the angles SPG, FPH are equal, and therefore PD bisects the angle SPF.

Hence $\quad SP : PF :: SD : DF :: SA : Ax,$

and P is a point in the curve.

Similarly P' is also a point in the curve, and the perpendicular from O, the centre of the circle, on FPP' meets it in V, the middle point of the chord PP'.

Since $\quad SE : EF :: SA : Ax$

and $\quad SD : DF :: SA : Ax;$

$$\therefore SE-SD : DE :: SA : Ax,$$

or $\quad SO : OD :: SA : Ax,$

a relation analogous to

$$SC : AC :: SA : AX.$$

We have already shewn, for each conic, that the middle points of parallel chords lie in a straight line; the following article contains a proof of the theorem which includes all the three cases.

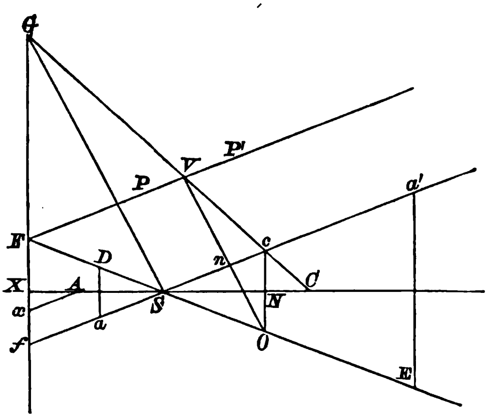
177. Prop. II. *To find the locus of the middle points of a system of parallel chords.*

Let $P'P$ one of the chords be produced to meet the directrix in F, draw Ax parallel to FP, and divide FS so that

$$SD : DF :: SE : EF :: SA : Ax;$$

then, as in the preceding article, the perpendicular OV upon PP' from O the middle point of DE, bisects PP'.

Draw the parallel focal chord aSa'; then Oc parallel to the directrix bisects aa' in c. Also draw SG perpendicular to the chords, and meeting the directrix in G.



Then, if OV meet aa' in n,

$$Vn : nO :: SF : SO,$$
$$:: Sf : Sc,$$

and, since ncO, SGf are similar triangles,

$$nO : nc :: SG : Sf;$$
$$\therefore\ Vn : nc :: SG : Sc,$$

and the line Vc passes through G.

The straight line Gc is therefore the locus of the middle points of all chords parallel to aSa'.

The ends of the diameter GC may be found by the construction of the preceding article.

Cor. When the conic is a parabola, $SA = AX$,

and $\quad Sa : af :: AX : Ax$

$\quad :: SX : Sf.$

So $\quad Sa' : a'f :: SX : Sf;$

$\therefore Sc : ac :: SX : Sf,$

and $\quad ac : cf :: SX : Sf.$

Hence $\quad Sc : cf :: SX^2 : Sf^2$

$\quad :: GX . Xf : Gf . fX$

$\quad :: GX : Gf;$

and therefore Gc is parallel to SX, that is, the middle points of parallel chords of a parabola lie in a straight line parallel to the axis.

178. **Prop. III.** *To find the locus of the middle points of all focal chords of a conic.*

Taking the case of a central conic, and referring to the figure of the preceding article, let Oc meet SC in N;

then $\quad CN : NS :: fX : SX,$

and $\quad CN : NC :: GX : CX;$

$\therefore cN^2 : SN . NC :: fX . GX : SX . CX$

$\quad :: SX^2 : SX . CX.$

Hence it follows that the locus of c is an ellipse of which SC is the transverse axis, and such that the squares of its axes are as $SX : CX$ or Cor. Art. (59), as

$$BC^2 : AC^2.$$

Hence the locus of c is similar to the conic itself.

EXAMPLES.

1. Given a focus, the corresponding directrix, and a tangent of a conic, find the point of contact.

2. If PK be the perpendicular from a point P of a conic on the directrix, and SK meet the tangent at the vertex in E, shew that the angles SPE, KPE are equal.

3. If F be any point in the directrix, and E, E' points on the directrix such that $FE=FE'=FS$, prove that EA and $E'A$ will meet the conic in the points of contact of tangents from F.

4. The portion of any tangent to a conic, intercepted between two fixed tangents, subtends a constant angle at the focus.

5. The focal distance of any point on a conic is equal to the length of the ordinate at the point produced to meet the tangent at the extremity of the latus rectum.

6. Prove that the normal at any point of a conic bears to the semi-latus rectum the ratio of the focal distance of the point of contact to the distance of the focus from the tangent.

7. If the tangent at the end of the latus rectum meet the tangent at the vertex, A, in T,

$$AT=AS.$$

8. If an ordinate, PNP', to the transverse axis meet the tangent at the end of the latus rectum in T,

$$TP \cdot TP'=SN^2.$$

Shew also that $SP=TN$.

9. SR being the semi-latus rectum, if RA meet the directrix in E, and SE meet the tangent at the vertex in T,

$$AT=AS.$$

10. A focal chord PSQ of a conic section is produced to meet the directrix in K, and KM, KN are drawn through the feet of the ordinates PM, QN of P and Q. If KN produced meet PM produced in R, prove that $PR=PM$.

11. Focal chords of an ellipse or hyperbola are in the ratio of the squares on the parallel diameters.

12. The tangents at P and Q, two points in a conic, intersect in T; if through P, Q, chords be drawn parallel to the tangents at Q and P, and intersecting the conic in p and q respectively, and if tangents at p and q meet in T, shew that Tt is a diameter.

13. From the foot of the directrix a line is drawn perpendicular to the tangent at P and meeting SP in R; the locus of R is a circle.

14. The chord of a conic PP' meets the directrix in K, and the tangents at P and P' meet in T; if RKR', parallel to ST, meet the tangents in R and R',

$$KR = KR'.$$

15. The tangents at P and P', intersecting in T, meet the latus rectum in D and D'; prove that the lines through D and D', respectively perpendicular to SP and SP', intersect in ST.

16. The chords PQ, $P'Q$ meet the directrix in F and F'; prove that the angle FSF' is half the angle between SP and SP'.

17. SPP' is a focal chord of an hyperbola; shew that any line through S is divided harmonically by the directrix and the tangents at P and P'.

18. Having given a focus, a tangent, and the eccentricity of a conic section; prove that the locus of its centre is a circle.

19. If the chord PQ of a conic meet the directrix in D, shew that SP, SQ are equally inclined to SD.

20. If P, Q be two points on a conic, and p, q two points on the directrix such that pq subtends at the focus half the angle subtended by PQ, either Pp and Qq or Pq and Qp meet on the curve.

21. A chord PP' of a conic meets the directrix in F, and from any point T in PP', TLL' is drawn parallel to SF and meeting SP, SP' in L and L'; prove that the ratio of SL or SL' to the distance of T from the directrix is equal to the ratio of $SA : AX$.

22. If an ellipse and hyperbola have their axes coincident and proportional, points on them equidistant from one axis have the sum of the squares on their distances from the other axis constant.

23. If PG be the normal at any point P of a conic, and GL the perpendicular from G upon SP, prove that

$$GL : PN :: SA : AX,$$

PN being the ordinate.

24. If Q be any point in the normal PG, QR the perpendicular on SP, and QM the perpendicular on PN,

$$QR : PM :: SA : AX.$$

25. If normals be drawn at the ends of a focal chord, a line through their intersection parallel to the axis of the conic will bisect the chord.

26. The normals at the extremities of a focal chord PSQ of a conic intersect in K, and KL is drawn perpendicular to PQ; KF is a diagonal of the parallelogram of which SK, KL, are adjacent sides: prove that KF is parallel to the transverse axis of the conic.

27. Given a focus of a conic section inscribed in a triangle, find the points where it touches the sides.

28. PSQ is any focal chord of a conic section; the normals at P and Q intersect in K, and KN is drawn perpendicular to PQ, prove that PN is equal to SQ, and hence deduce the locus of N.

29. If a chord PQ subtend a constant angle at the focus of a conic, the locus of the intersection of tangents at P and Q is a conic with the same focus and directrix.

30. If PSQ be a focal chord, and X the foot of the directrix, shew that XP, XQ, are equally inclined to the axis.

31. From the extremity P, of the diameter PQ of an ellipse, the tangent TPT' is drawn meeting two conjugate diameters in T, T'. From P, Q the lines PR, QR are drawn parallel to the same conjugate diameters. Prove that the rectangle under the semiaxes of the ellipse is a mean proportional between the triangles PQR and CTT'.

32. Shew that a conic may be drawn touching the sides of a triangle, having one focus at the centre of the circumscribing circle, and the other at the orthocentre.

33. The perpendicular from the focus of a conic on any tangent, and the central radius to the point of contact, intersect on the directrix.

34. From any given point in the axis of a conic a line is drawn perpendicular to the tangent at P; prove that the locus of its intersection with the focal distance of P is a circle.

35. A circle is described on the latus rectum of a conic through the focus S as diameter, and the common tangent to the conic and circle touches the conic in P and the circle in Q; prove that the angle PSQ is bisected by the latus rectum.

36. AB, AC are tangents to a conic at B, and C, and $DEGF$ is drawn from a point D in AC, parallel to AB and cutting the curve in E and F, and BC in G; shew that

$$DG^2 = DE \,.\, DF.$$

37. A diameter of a parabola, vertex F, meets two tangents in D and E and their chord of contact in G, shew that

$$FG^2 = FD \,.\, FE.$$

38. If a straight line parallel to an asymptote of an hyperbola meet the curve in F and also meet two tangents in D and E and their chords of contact in G;

$$FG^2 = FD \,.\, FE.$$

39. P and Q are two fixed points in a parabola, and from any other point R in the curve, RP, RQ are drawn cutting a fixed diameter, vertex E, in B and C; prove that the ratio of EB to EC is constant.

The same theorem is also true for an hyperbola, if a fixed line parallel to an asymptote be substituted for the diameter of the parabola.

CHAPTER X.

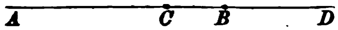
HARMONIC PROPERTIES, POLES AND POLARS.

179. Def. *A straight line is harmonically divided in two points when the whole line is to one of the extreme parts as the other extreme part is to the middle part.*

Thus AD is harmonically divided in C and B, when

$$AD : AC :: BD : BC.$$

This definition, it will be seen, is the same as that of Art. (17), for $BD=AD-AB$, and $BC=AB-AC$.



Under these circumstances the four points A, C, B, D constitute *an Harmonic Range*, and if through any point O four straight lines OA, OC, OB, OD be drawn, these four lines constitute an *Harmonic Pencil.*

180. Prop. I. *If a straight line be drawn parallel to one of the rays of an harmonic pencil, its segments made by the other three will be equal, and any straight line is divided harmonically by the four rays.*

Let $ACBD$ be the given harmonic range, and draw ECF through C parallel to OD, and meeting OA, OB in E and F.

Then $\quad AD : AC :: OD : EC,$

and $\quad BD : BC :: OD : CF;$

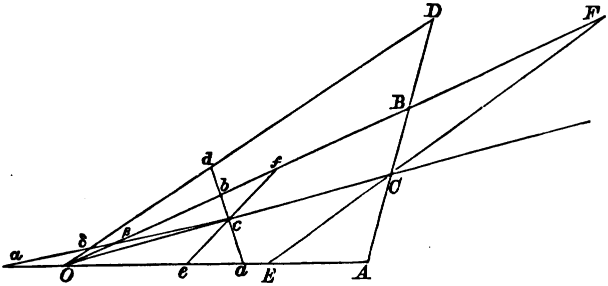
but from the definition

$$AD : AC :: BD : BC;$$

$$\therefore EC=CF,$$

and any other line parallel to *ECF* is obviously bisected by *OC*.

Next, let *acbd* be any straight line cutting the pencil, and draw *ecf* parallel to *Od*; so that $ec = cf$.



Then $ad : ac :: Od : ec$,

and $bd : bc :: Od : cf$;

$\therefore\ ad : ac :: bd : bc$;

that is, *acbd* is harmonically divided.

If the line $c\beta\delta a$ be drawn cutting *AO* produced,

then $a\delta : ac :: O\delta : ec$,

and $\beta\delta : \beta c :: O\delta : cf$;

$\therefore\ a\delta : ac :: \beta\delta : \beta c$,

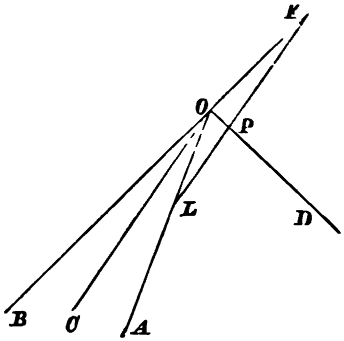
or $ac : a\delta :: \beta c : \beta\delta$,

and similarly it may be shewn in all other cases that the line is harmonically divided.

181. Prop. II. *The pencil formed by two straight lines and the bisectors of the angles between them is an harmonic pencil.*

For, if *OA*, *OB* be the lines, and *OC*, *OD* the bisectors, draw *KPL* parallel to *OC* and meeting *OA*, *OD*,

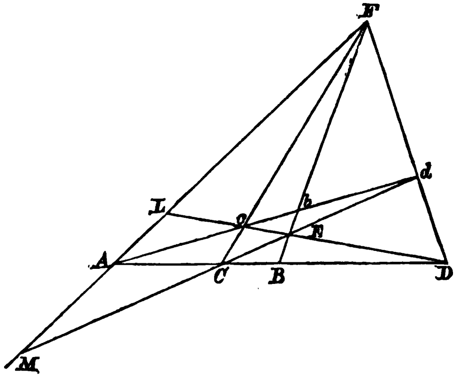
OB. Then the angles OKL, OLK are obviously equal,



and the angles at P are right angles; therefore $KP = PL$, and the pencil is harmonic.

182. Prop. III. *If* $ACBD$, $Acbd$ *be harmonic ranges, the straight lines* Cc, Bb, Dd *will meet in a point, as also* Cd, cD, Bb.

For, if Cc, Dd meet in F, join Fb; then the pencil



$F(Acbd)$ is harmonic, and will be cut harmonically by AD;

Hence Fb produced will pass through B.

Similarly, if Cd, cD meet in E,

$E(Acbd)$ is harmonic, and therefore bE produced will pass through B.

Harmonic Properties of a Quadrilateral.

In the preceding figure, let $CcdD$ be any quadrilateral; and let dc, DC meet in A, Cd, cD in E, and Cc, Dd in F.

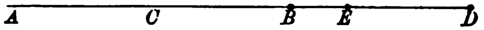
Then taking b and B so as to divide Acd and ACD harmonically, the ranges $Acbd$ and $ACBD$ are harmonic, and therefore Bb passes through both E and F.

Similarly it can be shewn that AF is divided harmonically in L and M, by Dc and dC.

For $E(Acbd)$ is harmonic and therefore the transversal $ALFM$ is harmonically divided.

183. Prop. IV. *If $ACBD$ be an harmonic range, and E the middle point of CD,*

$$EA \,.\, EB = EC^2.$$



For $\quad AD : AC :: BD : BC,$

or $\quad AE+EC : AE-EC :: EC+EB : EC-EB;$

$\quad \therefore AE : EC :: EC : EB,$

or $\quad AE \,.\, EB = EC^2 = ED^2.$

Hence also, conversely, if $EC^2 = ED^2 = AE \,.\, EB$, the range $ACBD$ is harmonic, C and D being on opposite sides of E.

Hence, if a series of points A, a, B, b,... on a straight line be such that

$$EA \,.\, Ea = EB \,.\, Eb = EC \,.\, Ec \ldots$$
$$= EP^2,$$

and if $EQ=EP$, then the several ranges $(APaQ)$, $(BPbQ)$, &c. are harmonic.

184. **Def.** A system of pairs of points on a straight line such that

$$EA\,.\,Ea=EB\,.\,Eb=\ldots\qquad=EP^2=EQ^2,$$

is called a system in *Involution*, the point E being called the centre and P, Q the foci of the system.

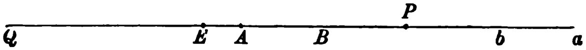
Any two corresponding points A, a, are called *conjugate points*, and it appears from above that any two conjugate points form, with the foci of the system, an harmonic range.

It will be noticed that a focus is a point at which conjugate points coincide, and that the existence of a focus is only possible when the points A and a are both on the same side of the centre.

185. **Prop. V.** *Having given two pairs of points A and a, B and b, it is required to find the centre and foci of the involution.*

If E be the centre,

$$EA : EB :: Eb : Ea;$$



$$\therefore\ EA : AB :: Eb : ab,$$

or $$EA : Eb :: AB : ab.$$

This determines E, and the foci P and Q are given by the relations

$$EP^2=EQ^2=EA\,.\,Ea.$$

We shall however find the following relation useful.

Since $$EA : Eb :: EB : Ea;$$

$$\therefore\ EA : Ab :: EB : aB,$$

or $$EA : EB :: Ab : aB;$$

but $$Eb : EA :: ab : AB;$$

$$\therefore\ Eb : EB :: Ab\,.\,ba :: AB\,.\,Ba.$$

Again, $$Qb : Pb :: QB : PB;$$

$$\therefore\ Qb - Pb : Pb :: QB - PB : PB,$$

or $$2\,.\,EP : Pb :: 2\,.\,EB : BP;$$

$$\therefore\ Pb^2 : PB^2 :: EP^2 : EB^2,$$

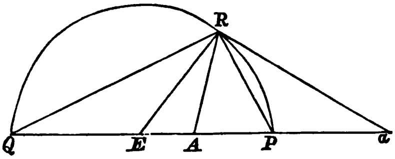
$$:: Eb : EB$$

$$:: Ab\,.\,ba : AB\,.\,Ba.$$

This determines the ratio in which Bb is divided by P.

186. If $QAPa$ be an harmonic range and E the middle point of PQ, and if a circle be described on PQ as diameter, the lines joining any point R on this circle with P and Q will bisect the angles between AR and aR.

For $$EA\,.\,Ea = EP^2 = ER^2;$$



$$\therefore\ EA : ER :: ER : Ea,$$

and the triangles ARE, aRE are similar.

Hence $$AR : aR :: EA : ER$$

$$:: EA : EP.$$

But $$Ea : EP :: EP : EA;$$

$$\therefore\ aP : EP :: AP : EA.$$

Hence $$AR : aR :: AP : aP,$$

and ARa is bisected by RP.

Hence, if A and a, B and b be conjugate points of a system in involution, of which P and Q are the foci, it follows that AB and ab subtend equal angles at any point of the circle on PQ as diameter.

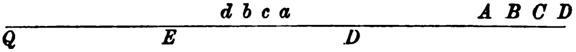
This fact also affords a means of obtaining the relations of Art. 185.

We must observe that if the points A, a are on one side of the centre and B, b on the other, the angles subtended by AB, ab are supplementary to each other.

187. Prop. VI. *If four points form an harmonic range, their conjugates also form an harmonic range.*

Let A, B, C, D be the four points,

a, b, c, d their conjugates.



Then, as in Art. 185,

$$Ea : ED :: ad : AD;$$

$$\therefore AD \, . \, Ea = ED \, . \, ad.$$

Similarly

$$\begin{aligned} AC \, . \, Ea &= EC \, . \, ac, \\ BD \, . \, Eb &= ED \, . \, bd, \\ BC \, . \, Eb &= EC \, . \, bc. \end{aligned}$$

But, $ABCD$ being harmonic,

$$AD : AC :: BD : BC;$$

$$\therefore ED \, . \, ad : EC \, . \, ac :: ED \, . \, bd : EC \, . \, bc.$$

Hence

$$ad : ac :: bd : bc,$$

or the range of the conjugates is harmonic.

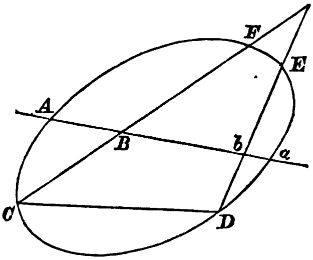
188. Prop. VII. *If a system of conics pass through four given points, any straight line will be cut by the system in a series of points in involution.*

Let the line meet one of the conics passing through the four points C, D, E, F, in A and a, and the sides CF, ED, in B and b.

Then the rectangles $AB \,.\, Ba$, $CB \,.\, BF$ are in the ratio of the squares on parallel diameters, as also are $Ab \,.\, ba$ and $Db \,.\, bE$.

But the squares on the diameters parallel to CF, ED are in the constant ratio $CB \,.\, BF : Db \,.\, bE$;

Therefore the rectangles $AB \,.\, Ba$, $Ab \,.\, ba$ are in a constant ratio.



But, Art. 185, this ratio is the same as that of $PB^2 : Pb^2$, if P be a focus of the involution A, a, B, b.

Hence P is determined, and all the conics cut the line Bb in points which form with B, b a system in involution.

We may observe that the foci are the points of contact of the two conics which can be drawn through the four points touching the line, and that the centre is the intersection of the line with the conic which has one of its asymptotes parallel to the line.

189. Prop. VIII. *If through any point two tangents be drawn to a conic, any other straight line through the point will be divided harmonically by the curve and the chord of contact.*

Let AB, AC be the tangents, $ADFE$ the straight line.

Through D and E draw $GDHK$, $LEMN$ parallel to BC.

Then the diameter through A bisects DH, and BC, and therefore bisects GK; hence $GD = HK$, and similarly $LE = MN$.

Also $\qquad LE : EN :: GD : DK$;

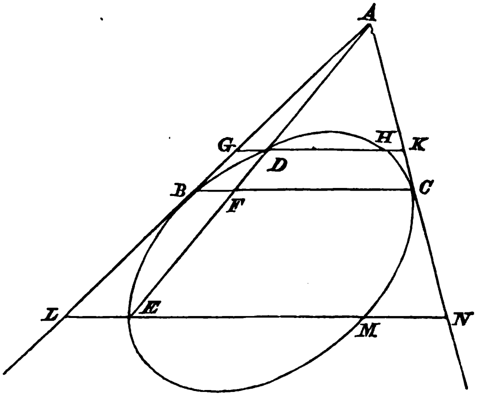
$\qquad \therefore LE . EN : LE^2 :: GD . DK : GD^2$,

or $\qquad LE . LM : GD . GH :: LE^2 : GD^2$

$\qquad\qquad :: LA^2 : GA^2$.

But $\qquad LE . LM : GD . GH :: LB^2 : BG^2$;

hence $\qquad AL : AG :: BL : BG$,

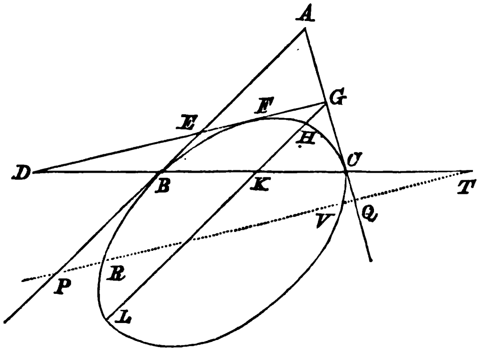


and therefore $\qquad AE : AD :: FE : FD$,

that is, $ADFE$ is harmonically divided.

190. Prop. IX. *If two tangents be drawn to a conic, any third tangent is harmonically divided by the two tangents, the curve, and the chord of contact.*

Let $DEFG$ be the third tangent, and through G, the point in which it meets AC, draw $GHKL$ parallel to AB, cutting the curve and the chord of contact in H, K, L.



Then $GH.GL : GC^2 :: AB^2 : AC^2$

$:: GK^2 : GC^2$;

$\therefore GH.GL = GK^2$.

Hence $DG^2 : DE^2 :: GK^2 : EB^2$

$:: GH.GL : EB^2$

$:: GF^2 : EF^2$;

that is, $DEFG$ is an harmonic range.

191. Prop. X. *If any straight line meet two tangents to a conic in P and Q, the chord of contact in T and the conic in R and V,*

$$PR.PV : QR.QV :: PT^2 : QT^2.$$

Taking the preceding figure, draw the tangent $DEFG$ parallel to PQ.

Then $PR.PV : EF^2 :: PB^2 : BE^2$

$:: PT^2 : DE^2$;

and $QR.QV : GF^2 :: QC^2 : GC^2$

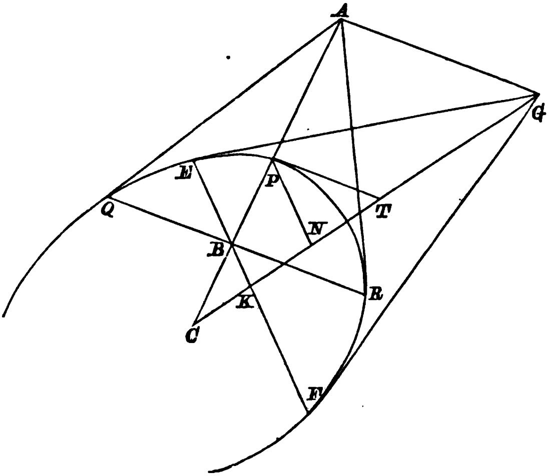
$:: QT^2 : DG^2$;

but $EF : DE :: GF : DG$;

$\therefore PR.PV : PT^2 :: QR.QV : QT^2$.

192. **Prop. XI.** ***If chords of a conic be drawn through a fixed point the pairs of tangents at their extremities will intersect in a fixed line.***

Let B be the fixed point and C the centre, and let CB meet the curve in P.



Take A in CP such that

$$CA : CP :: CP : CB;$$

then B is the middle point of the chord of contact of the tangents AQ, AR.

Draw any chord EBF and let the tangents at E and F meet in G: also join CG and draw PN parallel to EF.

Then if CG meet EF in K and the tangent at P in T,

$$CK . CG = CN . CT;$$

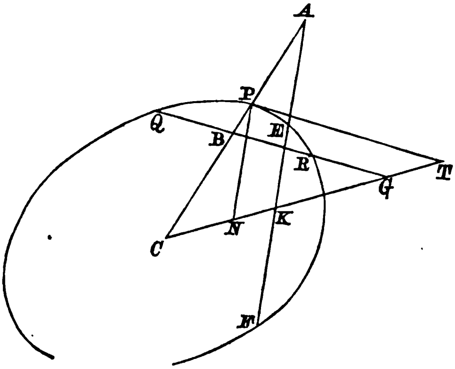
$$\begin{aligned}\therefore CG : CT &:: CN : CK\\ &:: CP : CB\\ &:: CA : CP;\end{aligned}$$

hence AG is parallel to PT, and the point G therefore lies on a fixed line.

If the conic be a parabola, we must take AP equal to BP: then, remembering that KG and NT are bisected by the curve, the proof is the same.

193. If A be the fixed point, and if the chord AEF be drawn through A, then, as before,

$$CK \, . \, CG = CN \, . \, CT,$$



and

$$\begin{aligned} CG : CT &:: CN : CK \\ &:: CP : CA \\ &:: CB : CP; \end{aligned}$$

therefore BG is parallel to PT and coincides with the chord of contact QR.

Hence, conversely, if from points on a straight line pairs of tangents be drawn to a conic, the chords of contact will pass through a fixed point.

194. Def. The locus of the points of intersection of tangents at the extremities of chords through a fixed point is called the *polar* of the point.

Also, if from points in a straight line pairs of tangents be drawn to a conic, the point in which all the chords of contact intersect is called the *pole* of the line.

If the pole be without the curve the polar is the chord of contact of tangents from the pole.

If the pole be on the curve, the polar is the tangent at the point.

195. PROP. XII. *A straight line drawn through any point is divided harmonically by the point, the curve, and the polar of the point.*

If the point be without the conic this is already proved in Art. (189).

If it be within the conic, as B in the figure of Art. (192), then, drawing any chord $FBEV$ meeting in V the polar of B, the chord of contact of tangents from V passes through B, by Art. (192), and the line $VEBF$ is therefore harmonically divided.

Hence the polar may be constructed by drawing two chords through the pole and dividing them harmonically; the line joining the points of division is the polar.

196. PROP. XIII. *The polars of two points intersect in the pole of the line joining the two points.*

For, if A, B be the two points and O the polar of AB, the line AO is divided harmonically by the curve, and therefore the polar of A passes through the point O.

Similarly the polar of B passes through O;

That is, the polars of A and B intersect in the pole of AB.

197. PROP. XIV. *If a quadrilateral be inscribed in a conic, its opposite sides and diagonals will intersect in three points such that each is the pole of the line joining the other two.*

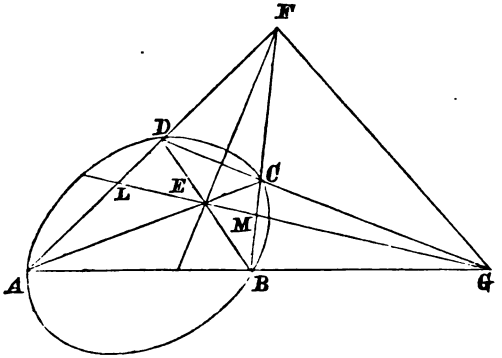
Let $ABCD$ be the quadrilateral, F and G the points of intersection of AD, BC, and of DC, AB.

Let EG meet FA, FB, in L and M.

Then, Art. (182), $FDLA$ and $FCMB$ are harmonic ranges;

Therefore L and M are both on the polar of F, Art. (195), and EG is the polar of F.

Similarly, EF is the polar of G, and therefore E is the pole of FG; Art. (196).



198. Def. If each of the sides of a triangle be the polar, with regard to a conic, of the opposite angular point, the triangle is said to be *self-conjugate* with regard to the conic.

Thus the triangle EGF in the above figure is self-conjugate.

To construct a self-conjugate triangle, take a straight line AB and find its polar C.

Draw through C any straight line CD cutting AB in D, and find the pole E of CD, which lies on AB: then CDE is self-conjugate.

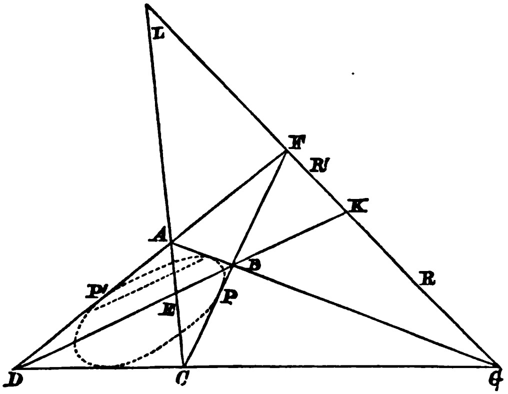
199. Prop. XV. *If a quadrilateral circumscribe a conic, its three diagonals form a self-conjugate triangle.*

If the chord of contact $P'P$ meet FG in R, R is on the polar of F, and therefore F is on the polar of R.

Now $F(AEBG)$ is harmonic;

$\therefore$ FE is the polar of R.

Similarly, if the other chord of contact meet FG in R', GE is the polar of R';



$\therefore$ E is the pole of FG.

Again, $DEBK$ is harmonic, and AB, AD are tangents; therefore AE passes through the polar of K; hence it follows that L, the intersection of AC, FG, is the pole of BD.

ELK is therefore a self-conjugate triangle.

200. Prop. XVI. *If a system of conics have a common self-conjugate triangle, any straight line passing through one of the angular points of the triangle is cut in a series of points in involution.*

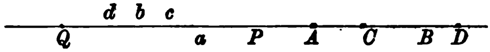
For, if ABC be the triangle, and a line $APDQ$ meet BC in D, and the conic in P and Q, $APDQ$ is an harmonic range, and all the pairs of points P, Q form with A and D an harmonic range.

Hence the pairs of points form a system in involution, of which A and D are the foci.

201. Prop. XVII. *The pencil formed by the polars of the four points of an harmonic range is an harmonic pencil.*

Let $ABCD$ be the range, O the polar of AD.

Let the polars Oa, Ob, Oc, Od meet AD in a, b, c, d, and let AD meet the conic in P and Q.



Then $APaQ$, $CPcQ$, &c. are harmonic ranges; and therefore, Arts. 183, 184, a, c, b, d are the conjugates of A, C, B, D.

Hence, Art. (187), the range $acbd$ is harmonic, and therefore the pencil $O\,(acbd)$ is harmonic.

Reciprocal Polars.

202. The pole of a line with regard to any conic being a point and the polar of a point a line, it follows that any system of points and lines can be transformed into a system of lines and points.

This process is called *reciprocation*, and it is clear that any theorem relating to the original system will have its analogue in the system formed by reciprocation.

Thus, if a series of lines be concurrent, the corresponding points are collinear; and the theorem of Art. (201) is an instance of the effect of reciprocation.

203. Def. If a point move in a curve (C), its polar will always touch some other curve (C'); this latter curve is called the reciprocal polar of (C) with regard to the auxiliary conic.

Prop. XVII. *If a curve C' be the polar of C, then will C be the polar of C'.*

For, if P, P' be two consecutive points of C, the intersection of the polars of P and P' is a point Q, which is the polar of the line PP'.

But the point Q is ultimately, when P and P' coincide, the point of contact of the curve which is touched by the polar of P.

Hence the polars of the points Q of C' are tangents to the curve C.

204. So far we have considered poles and polars generally with regard to any conic; we shall now confine our attention to the simple case in which a circle is the auxiliary curve.

In this case, if AB be a line, P its polar, and CY the perpendicular from the centre of the circle on AB, the rectangle $CP \, . \, CY$ is equal to the square on the radius of the circle.

A simple construction is thus given for the pole of a line, or the polar of a point.

205. As an illustration take the theorem of the existence of the orthocentre in a triangle.

Let AOD, BOE, COF be the perpendiculars, O being the orthocentre.

The polar reciprocal of the line BC is a point A', and of the point A a line $B'C'$.

To the line AD corresponds a point P on $B'C'$, and since ADB is a right angle, it follows that PSA' is a right angle, S being the centre of the auxiliary circle.

And, similarly, if SQ, SR, perpendiculars to SB', SC', meet $C'A'$ and $A'B'$ in Q and R, these points correspond to BE and CF.

But AD, BC, CF are concurrent,

$$\therefore P, Q, R \text{ are collinear.}$$

Hence the reciprocal Theorem,

If from any point S lines be drawn perpendicular respectively to SA', SB', SC', and meeting $B'C'$, $C'A'$, $A'B'$ in P, Q, and R, these points are collinear.

As a second illustration take the Theorem,

If A, B be two fixed points, and AC at right angles to each other, the locus of C is a circle.

Taking O, the middle point of AB, as the centre of the auxiliary circle, the reciprocals of A and B are two parallel straight lines, PE, QF, perpendicular to AB; the reciprocals of AC, BC are points P, Q on these lines such that POQ is a right angle, and PQ is the reciprocal of C.

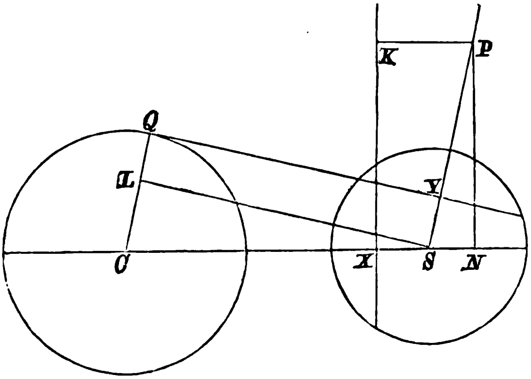
Hence, the locus of C being a circle, it follows that PQ always touches a circle.

The reciprocal Theorem therefore is,

If a straight line PQ, bounded by two parallel straight lines, subtend a right angle at a point O, halfway between the lines, the line PQ always touches a circle, having O for its centre.

206. Prop. XVIII. *The reciprocal polar of a circle with regard to another circle, called the auxiliary circle, is a conic, a focus of which is the centre of the auxiliary circle, and the corresponding directrix the polar of the centre of the reciprocated circle.*

Let S be the centre of the auxiliary circle, and KX the polar of C, the centre of the reciprocated circle.



Then, if P be the pole of a tangent QY to the circle C, SP meeting this tangent in Y,

$$SP \, . \, SY = SX \, . \, SC.$$

Therefore drawing SL parallel to QY,

$$SP : SC :: SX : QL.$$

But, by similar triangles,

$$SP : SC :: SN : CL;$$

$$\therefore SP : SC :: NX : CQ,$$

or $$SP : PK :: SC : CQ.$$

Hence the locus of P is a conic, focus S, directrix KX, and having for its eccentricity the ratio of SC to CQ.

The reciprocal polar of a circle is therefore an ellipse, parabola, or hyperbola, as the point S is within, upon, or without the circumference of the circle.

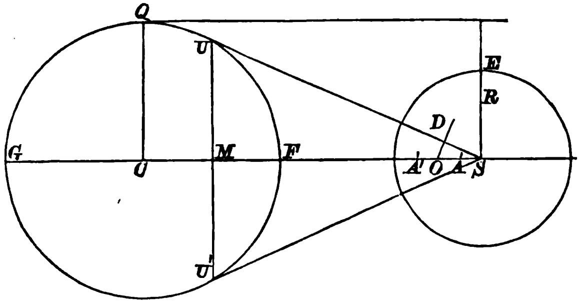
207. Prop. XIX. *To find the latus rectum and axes of the reciprocal conic.*

The ends of the latus rectum are the poles of the tangents parallel to SC.

Hence, if SR be the semi-latus rectum,

$$SR \,.\, CQ = SE^2,$$

SE being the radius of the auxiliary circle.



The ends of the transverse axis A, A' are the poles of the tangents at F and G;

$$\therefore SA \,.\, SG = SE^2,$$

and $$SA' \,.\, SF = SE^2.$$

Let SU, SU' be the tangents from S, then

$$SG . SF = SU^2,$$

$$\left.\begin{array}{ll} \therefore & SA' : SG :: SE^2 : SU^2, \\ \text{and} & SA : SF :: SE^2 : SU^2, \end{array}\right\} \ldots\ldots\ldots\ldots(a).$$

Hence $\quad AA' : FG :: SE^2 : SU^2,$

or, if O be the centre of the reciprocal,

$$AO : CQ :: SE^2 : SU^2.$$

Again, if BOB' be the conjugate axis,

$$BO^2 = SR . AO;$$

therefore, since $\quad SE^2 = SR . CQ,$

$$\begin{aligned} BO^2 : SE^2 &:: AO : CQ \\ &:: SE^2 : SU^2, \end{aligned}$$

and $\quad BO . SU = SE^2.$

The centre O, it may be remarked, is the pole of UU'.

For, from the relations (a),

$$\begin{aligned} SE^2 : SU^2 &:: SA + SA' : SF + SG \\ &:: SO : SC \\ &:: SO . SM : SC . SM; \end{aligned}$$

$$\therefore SO . SM = SE^2.$$

208. In the figures drawn, the reciprocal conic is an hyperbola; the asymptotes are therefore the lines through O perpendicular to SU and SU', the poles of these lines being at an infinite distance.

The semi-conjugate axis is equal to the perpendicular from the focus on the asymptote, Art. (99), *i.e.* if OD be the asymptote, SD is equal to the semi-conjugate axis.

Further, since OD is perpendicular to SU, and O is the pole of UU', it follows that D is the pole of CU, and

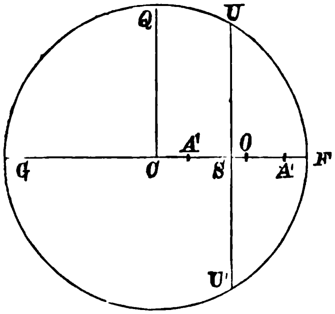
$$\therefore SD . SU = SE^2,$$

as we have already shewn.

Again, D being the intersection of the polars of C and U, is the intersection of SU and the directrix.

209. If the point S be within the circle, so that the reciprocal is an ellipse, the axes are given by similar relations.

Through S draw SU perpendicular to FG.



Then $$SF \, . \, SG = SU^2,$$

and exactly as in the case of the hyperbola,

$$AO : CQ :: SE^2 : SU^2,$$

and $$BO \, . \, SU = SE^2.$$

210. The important Theorem we have just considered enables us to deduce from any property of a circle, a corresponding property of a conic, and we are thus furnished with a method, which may serve to give easy proofs of known properties, or to reveal new properties of conics.

In the process of reciprocation we observe that points become lines and lines points; that a tangent to a curve reciprocates into a point on the reciprocal, that a curve inscribed in a triangle becomes a curve circumscribing a triangle, and that when the auxiliary curve is a circle, the reciprocal of a circle is a conic, the latus rectum of which varies inversely as the radius of the circle.

211. We give a few instances;

THEOREM.	RECIPROCAL.
The angles in the same segment of a circle are equal.	If a moveable tangent of a conic meet two fixed tangents, the intercepted portion subtends a constant angle at the focus.
Two of the common tangents of two equal circles are parallel.	If two conics have the same focus, and equal latera recta, the straight line joining two of their common points passes through the focus.
If a chord of a circle subtend a constant angle at a fixed point on the curve, the chord always touches a circle.	If two tangents of a conic move so that the intercepted portion of a fixed tangent subtends a constant angle at the focus, the locus of the intersection of the moving tangents is a conic having the same focus and directrix.
If a chord of a circle pass through a fixed point, the rectangle contained by the segments is constant.	The rectangle contained by the perpendiculars from the focus on two parallel tangents is constant.
If two chords be drawn from a fixed point on a circle at right angles to each other, the line joining their ends passes through the centre.	If two tangents of a conic move so that the intercepted portion of a fixed tangent subtends a right angle at the focus, the two moveable tangents meet in the directrix.
If a circle be inscribed in a triangle, the lines joining the vertices with the points of contact meet in a point.	If a triangle be inscribed in a conic the tangents at the vertices meet the opposite sides in three points lying in a straight line.
The sum of the reciprocals of the radii of the escribed circles of a triangle is equal to the reciprocal of the inscribed circle.	With a given point as focus, four conics can be drawn circumscribing a triangle, and the latus rectum of one is equal to the sum of the latera recta of the other three.

EXAMPLES.

1. If a series of circles pass through the same two points, any transversal will be cut by the circles in a series of points in involution.

2. If O be the centre of the circle circumscribing a triangle ABC, and $B'C'$, $C'A'$, $A'B'$, the respective polars with regard to a concentric circle of the points A, B, C, prove that O is the centre of the circle inscribed in the triangle $A'B'C'$.

3. OA, OB, OC being three straight lines given in position, shew that there are three other straight lines each of which forms with OA, OB, OC an harmonic pencil; and that each of the three OA, OB, OC forms with the second three an harmonic pencil.

4. The straight line $ABCD$ is divided harmonically in the points B, C; prove that if a circle be described on AC as diameter, any circle passing through B and D will cut it at right angles.

5. Three straight lines AD, AE, AF are drawn through a fixed point A, and fixed points B, C, D are taken in AD, such that $ABCD$ is an harmonic range. Any straight line through C intersects AE and AF in E and F, and BE, DF intersect in P; DE, BF in Q. Shew that P and Q always lie in a straight line through A, forming with AD, AE, AF an harmonic pencil.

6. CA, CB are two tangents to a conic section, O a fixed point in AB, POQ any chord of the conic; prove that the intersections of AP, BQ, and also of AQ, BP lie in a fixed straight line which forms with CA, CO, CB an harmonic pencil.

7. If three conics pass through the same four points, the common tangent to two of them is divided harmonically by the third.

8. Two conics intersect in four points, and through the intersection of two of their common chords a tangent is drawn to one of them; prove that it is divided harmonically by the other.

9. Prove that the two tangents through any point to a conic, any line through the point and the line to the pole of the last line, form an harmonic pencil.

10. Shew that the asymptotes of an hyperbola form, with any pair of conjugate diameters, an harmonic pencil.

11. Shew, from Arts. 195 and 196, that the centre of a conic is the pole of a line at an infinite distance.

12. PSQ and $PS'R$ are two focal chords of an ellipse; two other ellipses are described having P for a common focus, and touching the first ellipse at Q and R respectively. The three ellipses have equal major axes. Prove that the directrices of the last two ellipses pass through the pole of QR.

13. Tangents from T touch an ellipse in P and Q, and PQ meets the directrices in R and R'; shew that PR and QR' subtend equal angles at T.

14. If two confocal conics intersect, prove that the centre of curvature of either curve at a point of intersection is the pole of the tangent at that point with regard to the other curve.

15. If from a given point in the axis of a conic a chord be drawn, the perpendicular from the pole of the chord upon the chord will meet the axis in a fixed point.

16. Q is any point in the tangent at a point P of a conic; QG perpendicular to CP meets the normal at P in G, and QE perpendicular to the polar of Q meets the normal at P in E; prove that EG is constant and equal to the radius of curvature at P.

17. If any triangle be reciprocated with regard to its orthocentre, the reciprocal triangle will be similar and similarly situated to the original one and will have the same orthocentre.

18. If two conics have the same focus and directrix, and a focal chord be drawn, the four tangents at the points where it meets the conics intersect in the same point of the directrix.

19. An ellipse and a parabola have a common focus; prove that the ellipse either intersects the parabola in two points and has two common tangents with it, or else does not cut it.

20. Prove that the reciprocal polar of the circumscribed circle of a triangle with regard to the inscribed circle is an ellipse, the major axis of which is equal in length to the radius of the inscribed circle.

21. Prove that four parabolas, having a common focus, may be described so that each shall touch three out of four given straight lines.

22. A triangle ABC circumscribes a parabola, focus S; through ABC lines are drawn respectively perpendicular to SA, SB, SC; shew that these lines meet in a point.

23. A tangent to an ellipse at a point P intersects a fixed tangent in T; if through a focus S a line be drawn perpendicular to ST, meeting the tangent at P in Q, the locus of Q is a straight line touching the ellipse.

24. Prove that the distances, from the centre of a circle, of any two poles are to one another as their distances from the alternate polars.

25. If P, Q, R be three points on a conic, and PR, QR meet the directrix in p, q, the angle which pq subtends at the corresponding focus is half the angle which PQ subtends.

26. Reciprocate the theorems,

(1) The opposite angles of any quadrilateral inscribed in a circle are equal to two right angles.

(2) If a line be drawn from the focus of an ellipse making a constant angle with the tangent, the locus of its intersection with the tangent is a circle.

27. The locus of the intersection of two tangents to a parabola which include a constant angle is an hyperbola, having the same focus and directrix.

28. Two ellipses having a common focus cannot intersect in more than two real points, but two hyperbolas, or an ellipse and hyperbola, may do so.

29. ABC is any triangle and P any point: four conic sections are described with a given focus touching the sides of the triangles ABC, PBC, PCA, PAB respectively, shew that they all have a common tangent.

30. TP, TQ are tangents to a parabola cutting the directrix respectively in X and Y; ESF is a straight line drawn through the focus S perpendicular to ST, cutting TP, TQ respectively in E F; prove that the lines EY, XF are tangents to the parabola.

31. With the orthocentre of a triangle as focus, two conics are described touching a side of the triangle and having the other two sides as directrices respectively; shew that their minor axes are equal.

32. Prove that with a given point as focus, four conics can be drawn circumscribing a given triangle, and that the sum of the latera recta of three of them will equal the latus rectum of the fourth.

If the sides of the triangle subtend equal angles at the given point one of the conics will touch the other three.

33. Two parabolas have a common focus S; parallel tangents are drawn to them at P and Q intersecting the common tangent in P' and Q'; prove that the angle PSQ is equal to the angle between the axes, and the angle $P'SQ'$ is supplementary.

Deduce the reciprocal Theorem for two circles.

34. Shew that two circles can be reciprocated into a pair of confocal conics.

35. ABC is a given triangle, S a given point; on BC, CA, AB respectively, points A', B', C' are taken, such that each of the angles ASA', BSB', CSC', is a right angle. Prove that A', B', C' lie in the same straight line, and that the latera recta of the four conics, which have S for a common focus, and respectively touch the three sides of the triangles ABC, $AB'C'$, $A'BC'$, $A'B'C$ are equal to one another.

36. S is the focus of a conic; P, Q two points on it such that the angle PSQ is constant; through S, SR, ST are drawn meeting the tangents at P, Q in R, T respectively, and so that the angles PSR, QST are constant; shew that RT always touches a conic having the same focus and directrix as the original conic.

37. OA, OB are common tangents to two conics having a common focus S, CA, CB are tangents at one of their points of intersection, BD, AE tangents intersecting CA, CB in D, E. Prove that SDE is a straight line.

38. An hyperbola, of which S is one focus, touches the sides of a triangle ABC; the lines SA, SB, SC are drawn, and also lines SD, SE, SF respectively perpendicular to the former three lines, and meeting any tangent to the curve in D, E, F; shew that the lines AD, BE, CF are concurrent.

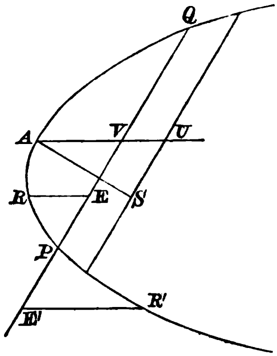
CHAPTER XI.

THE CONSTRUCTION OF A CONIC FROM GIVEN CONDITIONS.

212. It will be found that, in general, five conditions are sufficient to determine a conic, but it sometimes happens that two or more conics can be constructed which will satisfy the given conditions. We may have, as given conditions, points and tangents of the curve, the directions of axes or conjugate diameters, the position of the centre, or any characteristic or especial property of the curve.

213. Prop. I. *To construct a parabola, passing through three given points, and having the direction of its axis given.*

In this case the fact that the conic is a parabola is one of the conditions.



Let P, Q, R be the given points, and let RE parallel to the given direction meet PQ in E.

If E be the middle point of PQ, R is the vertex of the diameter RE; but, if not, bisecting PQ in V, draw the diameter through V and take A such that

$$AV : RE :: QV^2 : QE.EP.$$

Then A is the vertex of the diameter AV.

If the point E do not fall between P and Q, A must be taken on the side of PQ which is opposite to R.

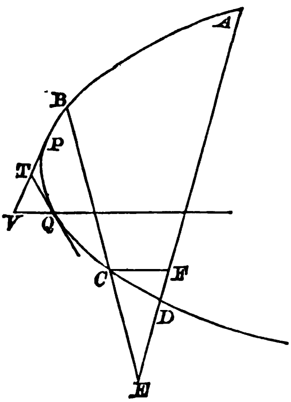
The focus may then be found by taking AU such that

$$QV^2 = 4AV.AU,$$

and by then drawing US parallel to QV and taking AS equal to AU.

214. PROP. II. *To describe a parabola through four given points.*

First, let $ABCD$ be four points in a given parabola, and let the diameter CF meet AD in F.



Draw the tangents PT, QT parallel to AD, BC, and the diameter QV meeting PT in V.

$$\text{Then}\quad ED.EA : EC.EB :: TP^2 : TQ^2$$
$$:: TV^2 : TQ^2$$
$$:: EF^2 : EC^2.$$

Hence the construction; in EA take EF such that

$$EF^2 : EC^2 :: ED.EA : EC.EB,$$

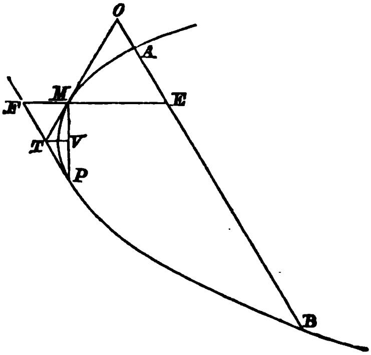
then CF is the direction of the axis, and the problem is reduced to the preceding.

If the point F be taken in AE produced, another parabola can be drawn, so that, in general, two parabolas can be drawn through four points.

215. This problem may be treated differently by help of the following theorem.

Prop. III. *If from a point O, outside a parabola, a tangent OM, and a chord OAB be drawn, and if the diameter ME meet the chord in E,*

$$OE^2 = OA.OB.$$



Let P be the point EM of contact of the tangent parallel to OAB, and let OM, ME meet this tangent in T and F.

Draw TV parallel to the axis and meeting PM in V;

then $$OA.OB : OM^2 :: TP^2 : TM^2$$
$$:: TF^2 : TM^2,$$

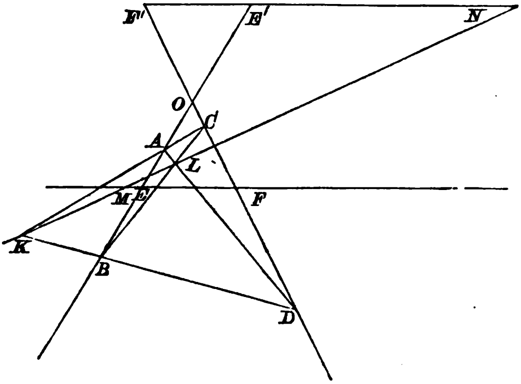
since PM is bisected in V;

also, $$TF : TM :: OE : OM;$$
$$\therefore OE^2 = OA.OB.$$

Hence, if A, B, C, D be the given points, and E, E', F, F', be so taken that

$$OE^2 = OE'^2 = OA.OB,$$

and $$OF^2 = OF'^2 = OC.OD,$$

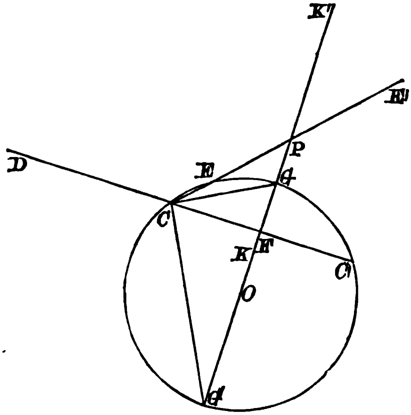


EF and $E'F'$ are diameters, and KL, the polar of O, will meet EF and $E'F'$ in M, N, the points of contact of tangents from O.

The second parabola is obtained by taking points in KAC, and KBD in the same manner.

216. Prop. IV. *Having given a pair of conjugate diameters, PCP', DCD', it is required to construct the ellipse.*

In CP take E such that $PE.PC = CD^2$, draw PF perpendicular to CD and take FC' equal to FC.



About CEC' describe a circle, cutting PF in G and G'; then

$$PG.PG' = PE.PC = CD^2,$$

and GCG' is a right angle; therefore CG and CG' are the directions of the axes and their lengths are given by the relations,

$$PG.PF = BC^2,$$

$$PG'.PF = AC^2.$$

We may observe that, O being the centre of the circle,

$$\begin{aligned} AC^2 + BC^2 &= PF.PG + PF.PG' \\ &= 2.PF.PO \\ &= 2.PC.PN, \end{aligned}$$

if N be the middle point of CE,

$$=PC^2+PC \cdot PE$$

$$=CP^2+CD^2.$$

If PE' be taken equal to PE in CP produced, and the same construction be made, we shall obtain the axes of an hyperbola having CP, CD for a pair of conjugate semi-diameters.

217. This problem may be treated also as follows.

In PF, the perpendicular on CD, take

$$PK=PK'=CD,$$

then $$PK^2=PG \cdot PG',$$

and therefore $K'GKG'$ is an harmonic range; and GCG' being a right angle, it follows, Art. 181, that CG and CG' are the bisectors of the angles between CK and CK'.

Hence, knowing CP and CD, G and G' are determined.

218. Prop. V. *Having given the focus and three points of a conic, to find the directrix.*

Let A, B, C, S be the three points and the focus.

Produce BA to D so that

$$BD : AD :: SB : SA,$$

and CB to E, so that

$$BE : CE :: SB : SC;$$

then DE is the directrix.

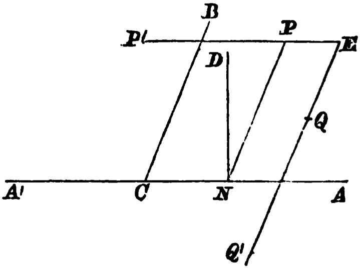
The lines BA, BC may be also divided internally in the same ratio, so that four solutions are generally possible.

Conversely, if three points A, B, C and the directrix are given, let BA, BC meet the directrix in D and E; then S lies on a circle, the locus of a point, the distances of which from A and B are in the ratio of AD to DB.

S lies also on a circle, similarly constructed with regard to BCE; the intersection of these circles gives two points either of which may be the focus.

219. **Prop. VI.** *Having given the centre, the directions of a pair of conjugate diameters, and two points of an ellipse, to describe the ellipse.*

If C be the centre, CA, CB the given directions, and P, Q the points, draw QMQ', PLP' parallel to CB and CA, and make $Q'M = QM$ and $P'L = PL$.



Then the ellipse will evidently pass through P' and Q', and if CA, CB be the conjugate radii, their ratio is given by the relation

$$CA^2 : CB^2 :: EP . EP' : EQ . EQ'.$$

Set up a straight line ND perpendicular to CA and such that

$$ND^2 : NP^2 :: EP . EP' : EQ . EQ',$$

and describe a circle, radius CD and centre C cutting CA in A, and take

$$CB : CA :: NP : ND.$$

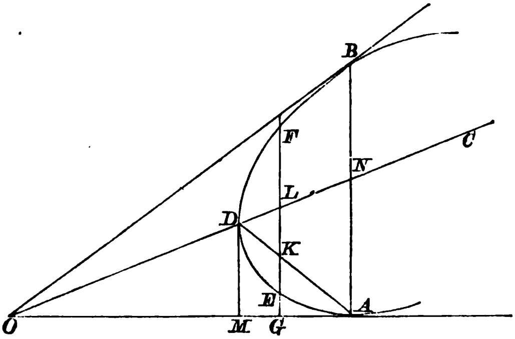
Then $$AN . NA' = ND^2,$$

and $$PN^2 : AN . NA' :: CB^2 : CA^2.$$

Hence CA, CB are determined, and the ellipse passes through P and Q.

220. **Prop. VII.** *To describe a conic passing through a given point and touching two given straight lines in given points.*

Let OA, OB be the given tangents, A and B the points of contact, N the middle point of AB.



1st. Let the given point D be in ON; then, if $ND = OD$, the curve is a parabola.

But if $ND < OD$, the curve is an ellipse, and, taking C such that $OC \,.\, CN = CD^2$, the point C is the centre.

If $ND > OD$ the curve is an hyperbola, and its centre is found in the same manner.

2nd. If the given point be E, not in ON, draw GEF, parallel to AB, and make FL equal to EL.

Take K such that

$$GK^2 = GE \,.\, EF,$$

then AK produced will meet ON in D, and the problem is reduced to the first case.

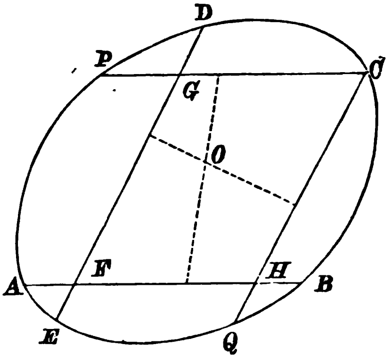
To justify this construction, observe that if DM be the tangent at D,

$$GE \,.\, GF : GA^2 :: DM^2 : DA^2$$
$$:: GK^2 : GA^2,$$

so that $$GE \,.\, GF = GK^2.$$

221. **Prop. VIII.** *To draw a conic through five given points.*

Let A, B, C, D, E be the five points, and F the intersection of DE, AB.



Draw CG, CH, parallel respectively to AB and CD and meeting CD, AB in G and H.

If F and G fall between D and E, and F and H between A and B, take GP in CG produced and HQ in CH produced, such that

$$CG.GP : DG.GE :: AF.FB : DF.FE,$$

and $CH.HQ : AH.HB :: DF.FE : AF.FB$;

Then, Arts. (86) and (129), P and Q are points in the conic.

Also PC, AB being parallel chords, the line joining their middle points is a diameter, and another diameter is obtained from CQ and DE.

If these diameters are parallel, the conic is a parabola, and we fall upon the case of Prop. II.; but if they intersect in a point O, this point is the centre of the conic, and, having the centre, the direction of a diameter, and two ordinates of that diameter, we fall upon the case of Prop. VI.

The figure is drawn for the case in which the pentagon *AEBCD* is not re-entering, in which case the conic may be any one of three.

If any one point fall within the quadrilateral formed by the other four, the curve is an hyperbola.

In all cases the points *P*, *Q* must be taken in accordance with the following rule.

The points *C*, *P*, or *C*, *Q* must be on the same or different sides of the points *G*, or *H*, according as the points *D*, *E*, or *B*, *A* are on the same or different sides of the points *G* or *H*.

Thus, if the point *E* be between *D* and *F*, and if *G* be between *D* and *E*, and *H* between *A* and *B*, the points *P* and *C* will be on the same side of *G*, and *C*, *Q* on the same side of *H*, but if *H* fall outside *A* and *B*, *C* and *Q* will be on opposite sides of *H*.

Remembering that if a straight line meet only one branch of an hyperbola, any parallel line will meet only one branch, and that if it meet both branches, any parallel will meet both branches, the rule may be established by an examination of the different cases.

222. The above construction depends only on the elementary properties of Conics, which are given in Chapters I, II, III, and IV. For some further constructions we shall adopt another method depending on harmonic properties.

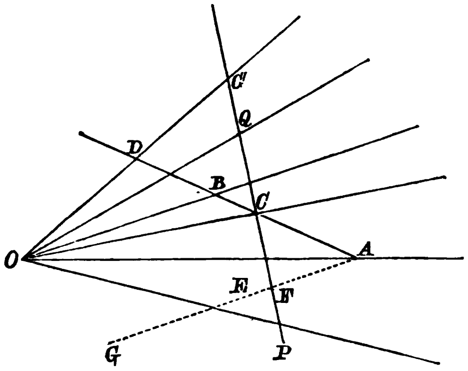
Prop. IX. *Having given two pairs of lines OA, OA′, and OB, OB′, to find a pair of lines OC, OC′, which shall make with each of the given pairs an harmonic pencil.*

This is at once effected by help of Art. (185).

For, if any transversal cut the lines in the points *c*, *a*, *b*, *c′*, *b′*, *a′*, the points *c*, *c′* are the foci of the involution, in which *a*, *a′* are conjugate, and also *b*, *b′*, the centre of the involution being the middle point of *cc′*.

223. Prop. X. *If two points and two tangents of a conic be given, the chord of contact intersects the given chord in one of two fixed points.*

Let OP, OQ be the given tangents, A and B the given points, and C the intersection of AB and the chord of contact.



Let OC' be the polar of C, and let AB meet OC in D,

Then C is on the polar of D, and therefore $DBCA$ is an harmonic range.

Also, C being on the polar of C', $C'QCP$ is an harmonic range.

Hence if two lines OC, OC' be found, which are harmonic with OA, OB, and also with OP, OQ, these lines intersect AB in two points C and D, through one of which the chord of contact must pass.

Or thus, if the tangents meet AB in a and b, find the foci C and D of the involution AB, ab; the chord of contact passes through one of these points.

224. **Prop. XI.** *Having given three points and two tangents, to find the chord of contact.*

In the preceding figure let OP, OQ be the tangents, and A, B, E the points.

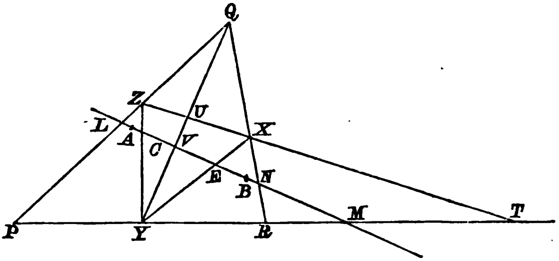
Find OC, OC' harmonic with OA, OB, and OP, OQ; also find OF, OG harmonic with OA, OE and OP, OQ.

Then any one of the four lines joining C or D, to F or G is a chord of contact, and the chord of contact and points of contact being known, the case reduces to that of Art. (220).

Hence four such conics can in general be described.

225. **Prop. XII.** *To describe a conic, passing through two given points, and touching three given straight lines.*

Let AB, the line joining the given points, meet the given tangents QR, RP, PQ, in N, M, L.



Find the foci C, D of the involution A, B and L, M;

Then the polar of P, YZ passes through C or D.

Also find the foci, F, G, of the involution A, B, and M, N; then the polar of R, XY, passes through F or G.

Let ZX meet PR in T; then T is on the polar of Q, and QY is the polar of T.

Hence $TXUZ$ is harmonic;

therefore $MFVC$ is harmonic.

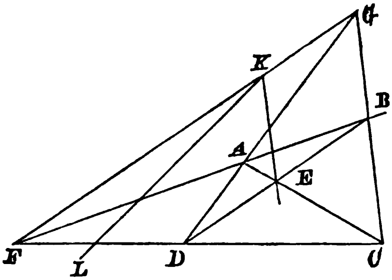
This determines V, and joining QV, we obtain the point of contact Y.

Then, joining YC and YF, Z and X are obtained.

Since either C or D may be taken with F or G, there are in general four solutions of the problem.

226. Prop. XIII. *To describe a conic, having given four points and one tangent.*

Let A, B, C, D be the given points, and complete the quadrilateral.



Then E is the pole of FG, and if the given tangent KL meet FG in K, E is on the polar of K; therefore the other tangent through K forms an harmonic pencil with KF, KL, KE.

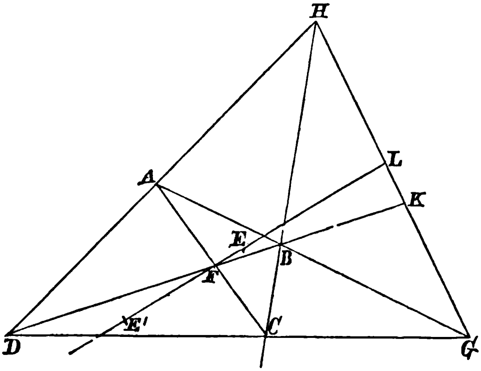
Hence two tangents being known, and a point E in the chord of contact, if we find two points P, P' in A, B, such that KP, KP' are harmonic with KA, KB, and also with KL, KL', we shall have two chords of contact EP, EP', and therefore two points of contact for KL and also for KL'.

Hence two conics can be described.

We observe that if two conics pass through four points, their common tangents meet on one of the sides of the self-conjugate triangle EFG.

227. Prop. XIV. *Given four tangents and one point, to construct the conic.*

Let *ABCD* be the given circumscribing quadrilateral, and *E* the given point. Completing the figure, draw *LEF* through *E* and *F*, and complete the harmonic range *LEFE'*; then, since *F* is the pole of *HG*, Art. (197), *E'* is a point in the conic.



Also since *K* is the pole of *FH*, Art. 199, the chord of contact of the tangents, *AB*, *AD* passes through *K*.

Hence the construction is the same as that of Art. 226, and there are two solutions of the problem.

228. Prop. XV. *Given five points, to construct the conic.*

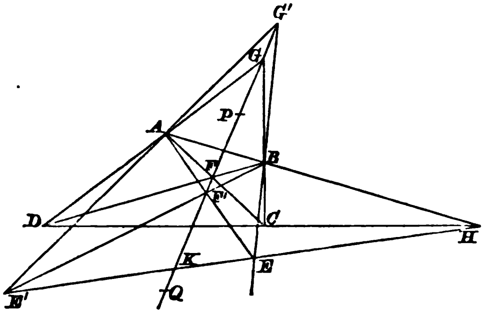
Let *A*, *B*, *C*, *D*, *E* be the five points, and complete the quadrilateral *ABCD*.

Then *H* is the polar of *FG*, and *FG* passes through the points of contact *P*, *Q* of the tangents from *H*.

Join *HE*, cutting *FG* in *K*, and complete the harmonic range *HEKE'*; then *E'* is a point in the conic.

Also *AE*, *BE* will intersect *FG* in the same point *F*, and *E'A*, *EB* will also intersect *FG* in the same point *G'*.

But $GPFQ$ and $G'PF'Q$ are both harmonic ranges; therefore P and Q are the foci of an involution of which F, G and F', G' are pairs of conjugate points.

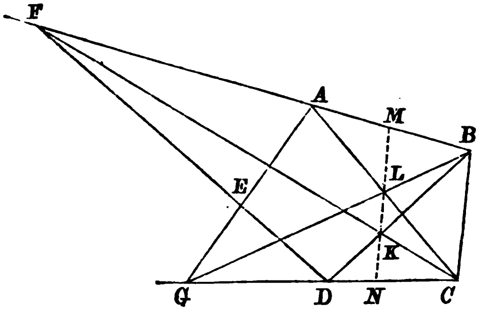


Hence, finding these foci, P and Q, the tangents HP, HQ are known, and the case is reduced to that of Prop. VII.

Hence only one conic can be drawn through five points.

229. Prop. XVI. *Given five tangents, to find the points of contact.*

Let $ABCDE$ be the circumscribing pentagon. Considering the quadrilateral $FBCD$, join FC, BD meeting in K.



Then, (Art. 199), K is the pole of the line joining the intersections of FB, CD, and of FD, BC; that is, the chords of contact of BF, CD, and of BC, FD meet in K.

Similarly if BG, AC meet in L, the chords of contact of AB, CG, and of BC, AG meet in L.

Hence KL is the chord of contact of AB, CD, and therefore determines M, N the points of contact.

Hence it will be seen that only one conic can be drawn touching five lines.

CHAPTER XII.

THE OBLIQUE CYLINDER, THE OBLIQUE CONE, AND THE CONOIDS.

230. **Def.** If a straight line, which is not perpendicular to the plane of a given circle, move parallel to itself, and always pass through the circumference of the circle, the surface generated is called an oblique cylinder.

The line through the centre of the circular base, parallel to the generating lines, is the axis of the cylinder.

It is evident that any section by a plane parallel to the axis consists of two parallel lines, and that any section by a plane parallel to the base is a circle.

The plane through the axis perpendicular to the base is the principal section.

The section of the cylinder by a plane perpendicular to the principal section, and inclined to the axis at the same angle as the base is called a subcontrary section.

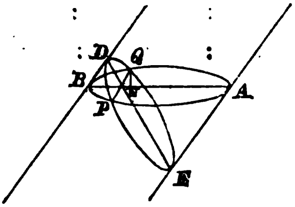
231. **Prop. I.** *The subcontrary section of an oblique cylinder is a circle.*

The plane of the paper being the principal plane and APB the circular base, a subcontrary section is DPE, the angles BAE, DEA being equal.

Let PQ be the line of intersection of the two circles;

then $PN.NQ$ or $PN^2 = BN.NA$.

But $NB = ND$, and $NA = NE$;

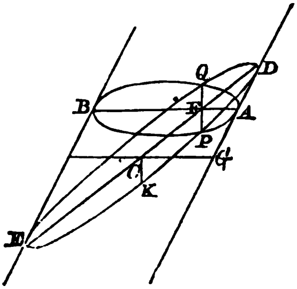


$$\therefore PN . NQ = DN . NE,$$

and DPE is a circle.

232. PROP. II. *The section of an oblique cylinder by a plane which is not parallel to the base or to a subcontrary section is an ellipse.*

Let the plane of the section, DPE, meet any circular section in the line PQ, and let AB be that diameter of the



circular section which is perpendicular to PQ, and bisect PQ in the point F.

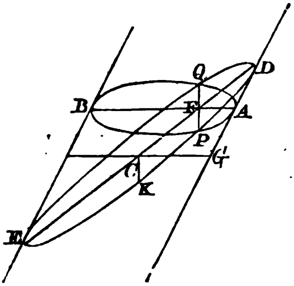
Let the plane through the axis and the line AB cut the section DPE in the line DFE.

Then $$PF^2 = AF . FB.$$

But if DE be bisected in C, and GKC be the circular section through C parallel to APB,

$$AF : FD :: CG : CD,$$

and
$$FB : FE :: CG : CD;$$



$$\therefore\ AF.FB : DF.FE :: CG^2 : CD^2;$$

hence, observing that $CG = CK$,

$$PF^2 : DF.FE :: CK^2 : CD^2.$$

But, if a series of parallel circular sections be drawn, PQ is always parallel to itself and bisected by DE;

Therefore the curve DPE is an ellipse, of which CD, CK are conjugate semi-diameters.

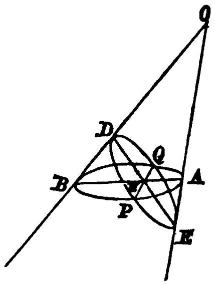
233. Def. If a straight line pass always through a fixed point and the circumference of a fixed circle, and if the fixed point be not in the straight line through the centre of the circle at right angles to its plane, the surface generated is called an oblique cone.

The plane containing the vertex and the centre of the base, and also perpendicular to the base is called the principal section,

The section made by a plane not parallel to the base, but perpendicular to the principal section, and inclined to the generating lines in that section at the same angles as the base, is called a subcontrary section.

234. Prop. III. *The subcontrary section of an oblique cone is a circle.*

The plane of the paper being the principal section, let



APB be parallel to the base and DPE a subcontrary section, so that the angle

$$ODE = OAB,$$

and $$OED = OBA.$$

The angles DBA, DEA being equal to each other, a circle can be drawn through $BDAE$.

Hence, if PNQ be the line of intersection of the two planes,

$$DN.NE = BN.NA,$$

$$= PN.NQ;$$

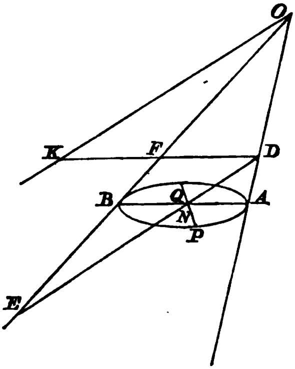
therefore DPE is a circle.

And all sections by planes parallel to DPE are circles.

Planes parallel to the base, or to a subcontrary section, are called also *Cyclic Planes.*

235. Prop. IV. *The section of a cone by a plane not parallel to a cyclic plane is an Ellipse, Parabola, or Hyperbola.*

(1) Let the section, DPE, meet all the generating lines on one side of the vertex.



Let any circular section cut DPE in PQ, and take AB the diameter of the circle which is at right angles to, and bisects PQ.

The plane OAB will cut the plane of the section in a line DNE.

Draw OK parallel to DE and meeting in K the plane of the circular section through D parallel to APB, and join DFK.

Then $AN : ND :: KD : OK$,

and $BN : NE :: KF : OK$;

therefore $AN.NB : DN.NE :: KD.KF : OK^2$,

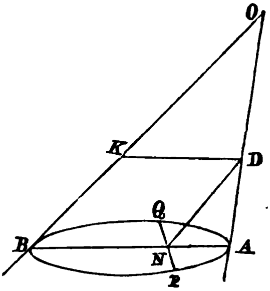
or $PN^2 : DN.NE :: KD.KF : OK^2$.

But if a series of circular sections be drawn the lines PQ will always be parallel, and bisected by DE;

Therefore the curve DPE is an ellipse, having DE for a diameter, and the conjugate diameter parallel to PQ, and the squares on these diameters are in the ratio of $KD.KF$ to OK^2.

(2) Let the section be parallel to a tangent plane of the cone.

If OB be the generating line along which the tangent plane touches the cone, and BT the tangent line at B to a circular section through B, the line of intersection PQ will be parallel to BT, and therefore perpendicular to the diameter BA through B.



Let the plane BOA cut the plane of the section in DN.

Then drawing DK parallel to AB,

$$BN = KD,$$

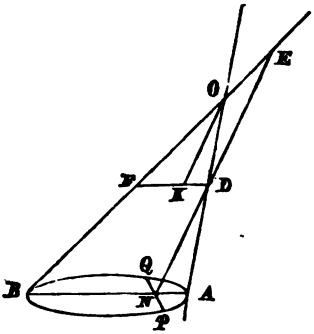
and $$AN : ND :: KD : OK;$$

therefore $$AN.NB : ND.KD :: KD : OK,$$

or $$PN^2 : ND.KD :: KD : OK,$$

and KD, OK being constant, the curve is a parabola having the tangent at D parallel to PQ.

If the plane of the section meet both branches of the cone, make the same construction as before, and we shall obtain, in the same manner as for the ellipse,



$$PN^2 : DN.NE :: DK.KF : OK^2,$$

OK being parallel to DE.

Therefore, since the point N is not between the points D and E, the curve DP is an hyperbola.

Conoids.

236. Def. *If a conic revolve about one of its principal axes, the surface generated is called a conoid.*

If the conic be a circle, the conoid is a sphere.

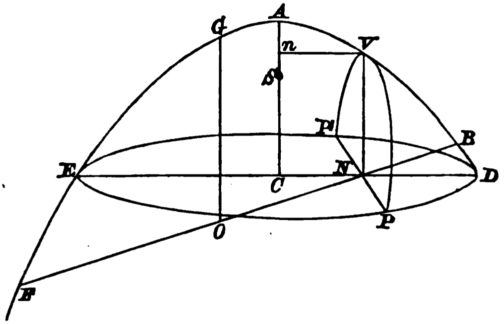
If the conic be an ellipse, the conoid is an oblate or a prolate spheroid according as the revolution takes place about the conjugate or the transverse axis.

If it be an hyperbola the surface is an hyperboloid of one or two sheets, according as the revolution takes place about the conjugate or transverse axis, and the surface generated by the asymptotes is called the asymptotic cone.

If the conic consist of two intersecting straight lines, the limiting form of an hyperbola, the revolution will be about one of the lines bisecting the angles between them, and the conoid will then be a right circular cone.

237. Prop. V. *A section of a paraboloid by a plane parallel to the axis is a parabola equal to the generating parabola, and any other section not perpendicular to the axis is an ellipse.*

Let PVN be a section parallel to the axis, and take



the plane of the paper perpendicular to the section and cutting it in VN.

Take any circular section DPE, cutting the section PVN in PNP'.

Then PN is perpendicular to DE,

and

$$\begin{aligned} PN^2 &= DN \,.\, NE \\ &= DC^2 - NC^2 \\ &= 4AS \,.\, Ac - 4AS \,.\, An \\ &= 4AS \,.\, VN; \end{aligned}$$

therefore the curve VP is a parabola equal to EAD.

Again, let BPF be a section not parallel or perpendicular to the axis, but perpendicular to the plane of the paper;

Then, $BN.NF=4SG.VN$, OG being the diameter bisecting BF;

therefore $PN^2 : BN.NF :: AS : SG$,

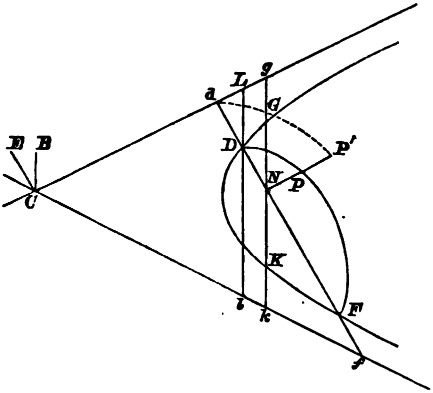
and the curve BPN is an ellipse.

Moreover if the plane BF move parallel to itself, SG is unaltered, and the *sections by parallel planes are similar ellipses.*

In exactly the same manner, it may be shewn that the oblique sections of spheroids are ellipses, and those of hyperboloids either ellipses or hyperbolas.

238. Prop. VI. *The sections of an hyperboloid and its asymptotic cone by a plane are similar curves.*

Taking the case of an hyperboloid of two sheets, let DPF, $dP'f$, be the sections of the hyperboloid and cone,



$P'PN$ the line in which their plane is cut by a circular section GPK or $gP'k$.

Through D draw LDl perpendicular to the axis; then, since $PN^2=GN.NK$, and $$P'N^2 = gN.Nk,$$

$$P'N^2 : dN.Nf :: gN.Nk : dN.Nf,$$
$$:: LD.lD : Dd.Df,$$
$$:: BC^2 : CE^2$$

if CE be the semidiameter parallel to DF;

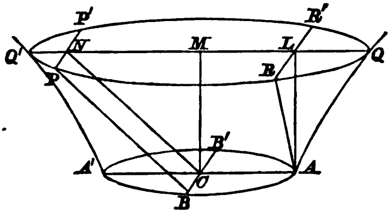
and $$PN^2 : DN.NF :: GN.NK : DN.NF$$
$$:: BC^2 : CE^2;$$

therefore the curves DPF, $dP'f$ have their axes in the same ratio, and are similar ellipses.

In the same manner the theorem can be established if the sections be hyperbolic, or if the hyperboloid be of one sheet.

239. Prop. VII. *If an hyperboloid of one sheet be cut by a tangent plane of the asymptotic cone, the section will consist of two parallel straight lines.*

Let AQ, $A'Q'$ be a section through the axis, CN the generating line, in the plane AQ, along which the tangent



plane touches the cone; and PNP' the section with this tangent plane of a circular section QPQ'.

Then $$PN^2 = QN.NQ'$$
$$= AC^2, \text{ Art. (102)};$$

therefore if BCB' be the diameter, perpendicular to the plane CAQ, of the principal circular section,

$$PN = BC \text{ and } P'N = B'C;$$

therefore PB and $P'B'$ are each parallel to CN; that is, the section consists of two parallel straight lines.

240. Prop. VIII. *The section of an hyperboloid of one sheet by a plane parallel to its axis, and touching the central circular section, consists of two straight lines.*

Let the plane pass through A, and be perpendicular to the radius CA of the central section; fig. Art. (239).

The plane will cut the circular section QPQ' in a line RLR', and

$$RL^2 = QL \,.\, LQ' = QM^2 - AC^2,$$

if M be the middle point of QQ'.

But $\quad QM^2 - AC^2 : CM^2 :: AC^2 : BC^2$;

therefore $\quad RL : AL :: AC : BC$;

hence it follows that AR is a fixed line; and similarly AR' is also a fixed line.

It will be seen that these lines are parallel to the section of the cone by the plane through the axis perpendicular to CA.

241. Prop. IX. *If a conoid be cut by a plane, and if spheres be inscribed in the conoid touching the plane, the points of contact of the spheres with the plane will be the foci of the section, and the lines of intersection of the planes of contact with the plane of section will be the directrices.*

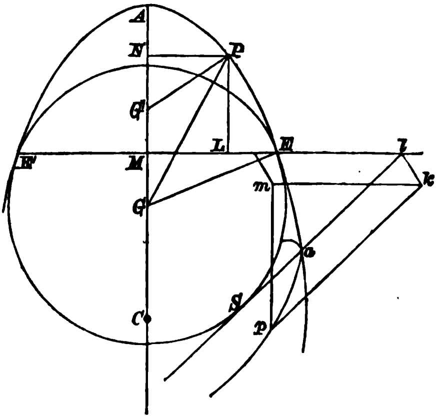
In order to establish this statement, we shall first demonstrate the following theorem;

If a circle touch a conic in two points, the tangent from any point of the conic to the circle bears a constant ratio to its distance from the chord of contact.

Take the case of an ellipse, the chord of contact being perpendicular to the transverse axis.

If EME' be this chord, the normal EG is the radius of the circle, and if PT be a tangent from a point P of the ellipse,

$$\begin{aligned} PT^2 &= PG^2 - GE^2 \\ &= PN^2 + NG^2 - EM^2 - MG^2, \end{aligned}$$



But $EM^2 - PN^2 : CN^2 - CM^2 :: BC^2 : AC^2$,

and $CN^2 - CM^2 = MN(CM + CN)$.

Let the normal at P meet the axis in G';

then $NG' : CN :: BC^2 : AC^2$,

and $MG : CM :: BC^2 : AC^2$;

therefore $NG' + MG : CN + CM :: BC^2 : AC^2$.

Hence $EM^2 - PN^2 = MN(NG' + MG)$.

Also $NG^2 - MG^2 = MN(NG + MG)$;

therefore

$$\begin{aligned} PT^2 &= MN(NG + MG) - MN(NG' + MG) \\ &= MN \,.\, GG'. \end{aligned}$$

But $CG : CM :: SC^2 : AC^2$,

and $$CG' : CN :: SC^2 : AC^2;$$

therefore $$GG' : MN :: NC^2 : AC^2.$$

Hence $$PT^2 : PL^2 :: SC^2 : AC^2,$$

PL being equal to MN.

This being established let the figure revolve round the axis AC, and let a plane section ap of the conoid, perpendicular to the plane of the paper, touch the sphere at S and cut the plane of contact EE' in lk.

From a point P of the section let fall the perpendicular pm on the plane EE', draw mk perpendicular to lk, and join pk.

Then $pm : pk$ is a constant ratio.

Also taking the meridian section through P, pS is equal to the tangent from p to the circular section of the sphere, and is therefore in a constant ratio to pm;

Hence Sp is to Pk in a constant ratio,

and therefore S is the focus and kl the directrix of the section ap.

242. If the curve be a parabola, the proof is as follows:

$$\begin{aligned} PT^2 &= PG^2 - EG^2 \\ &= PN^2 + NG^2 - EM^2 - MG^2 \\ &= MN(NG + MG) - 4AS\,.\,MN \\ &= MN(NG + MG) - 2MG\,.\,MN \\ &= MN^2. \end{aligned}$$

It will be found that the theorem is also true for an hyperboloid of two sheets, and for an hyperboloid of one sheet, but that in the latter case the constant ratio of PT to PL is not that of SC to AC.

243. The geometrical enunciation of the theorem also requires modification in several cases. To illustrate the difficulty, take the paraboloid, and observe that if the

normal at E cut the axis in G, and if O be the centre of curvature at A,

$$AG > AO,$$

and the radius of the circle is never less than AO.

This shews that a circle the radius of which is less than AO cannot be drawn so as to touch the conic in two points.

We may mention one exceptional case in which the theorem takes a simple form.

In general

$$EG^2 = EM^2 + MG^2 = 4AS(AM + AS)$$
$$= 4AS \, . \, SG.$$

Taking the point g between S and O, describe a circle centre g and such that the square on its radius $= 4AS \, . \, Sg$.

Also take a point F in the axis produced such that

$$AF = Og;$$

it will then be found that the tangent from P to the circle will be equal to NF.

When g coincides with S, the circle becomes a point,

and $$AF = AS;$$

we thus fall back on the fundamental definition of a parabola.

It will be found that if the plane section of the conoid pass through S, the point S is a focus of the section.

APPENDIX.

244. *It being given that PN^2 is to $AN.NA'$ in a constant ratio, and that N lies between A and A', we can deduce as follows the existence of a focus and directrix for the ellipse which is the locus of P.*

Bisect AA' in C, and draw CB perpendicular to AA', and of such a length that BC^2 is to AC^2 in the given ratio.

With centre B and radius AC describe a circle cutting AA' in S and S', and produce CA to X, so that

$$CX : CA :: CA : CS,$$

then $$CS.SX = CS.CX - SC^2 = AC^2 - SC^2 = CB^2.$$

Let PA, $A'P$ meet in E and F the line through X perpendicular to AA'; fig. Art. (55),

then $$\begin{aligned} EX.FX : AX.A'X &:: PN^2 : AN.NA' \\ &:: BC^2 : AC^2, \end{aligned}$$

or $$\begin{aligned} EX.FX : BC^2 &:: CX^2 - CA^2 : CA^2 \\ &:: CA^2 - CS^2 : CS^2 \\ &:: CB^2 : CS^2 \\ &:: SX^2 : CB^2; \end{aligned}$$

therefore $$EX.FX = SX^2,$$

and ESF is a right angle.

Let the perpendicular PK on FX meet FS in L and ES in l, fig. Art. (53);

then $$PL : PK :: SA' : A'X,$$
$$:: SA : AX,$$
$$:: Pl : PK;$$

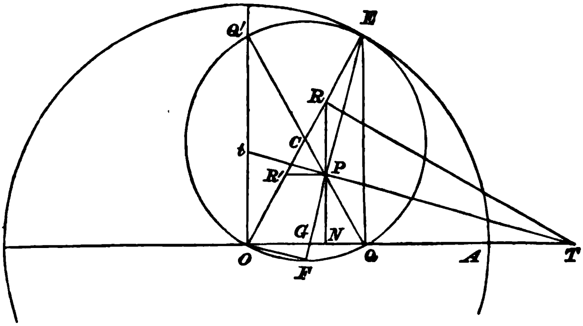
therefore $$PL = Pl = SP,$$

since LSl is a right angle,

and $$SP : PK :: S'A : AX.$$

245. *If a circle roll on the inside of the circumference of a circle of double its radius, any point in the area of the rolling circle traces out an ellipse.*

Let C be the centre of the rolling circle, E the point of contact.



Then, if the circle meet in Q a fixed radius OA of the fixed circle, the angle ECQ is twice the angle EOA and therefore the arcs EQ, EA are equal.

Hence, when the circles touch at A, the point Q of the rolling circle coincides with A, and the subsequent path of Q is the diameter through A.

Let P be a given point in the given radius CQ, and RPN perpendicular to OA;

Then, OQE being a right angle, EQ is parallel to RP and therefore $CR = CP$, and OR is constant.

Also $$PN : RN :: PQ : OR;$$

therefore, the locus of R being a circle, the locus of P is an ellipse, whose axes are as $PQ : OR$.

But OR is clearly the length of one semi-axis, and PQ or OR' is therefore the length of the other, OR, OR' being equal to $OC+CP$ and $OC-CP$.

246. Properties of the ellipse are deducible from this construction.

Thus, as the circle rolls, the point E is instantaneously at rest, and the motion of P is therefore at right angles to EP, *i. e.*, producing EP to F, in the direction FO.

Therefore, drawing PT parallel to OF, PT is the tangent, and PF the normal.

A circle can be drawn through $EPQT$ since EPT, EQT are right angles; but the circle through QPE clearly passes through R; therefore the angle ORT is a right angle, and

$$ON : OR :: OR : OT,$$

or $$ON . OT = OR^2,$$

a known property of the tangent.

Again if PF meet OQ in G, the angles PQG, PFQ are equal, being on equal bases EQ, OQ';

therefore $$PG : PQ :: PQ : PF,$$

or $$PG . PF = PQ^2 = OR'^2;$$

a known property of the normal.

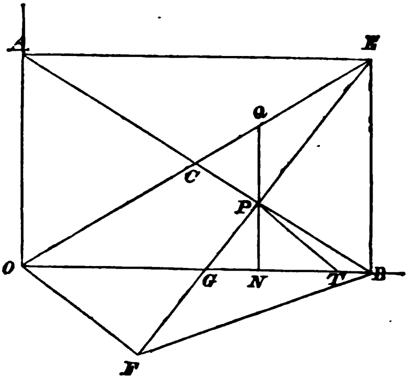
247. *A given straight line has its ends moveable on two straight lines at right angles to each other; the path of any given point in the moving line is an ellipse.*

Let P be the point in the moving line AB, and C the middle point of AB.

Let the ordinate NP, produced if necessary, meet OC in Q; then $CQ=CP$ and OQ is equal to AP, so that the locus of Q is a circle.

Also $PN : QN :: PB : OQ$

$$:: PB : PA;$$



therefore the locus of P is an ellipse, and its semi-axes are equal to AP and BP.

248. From this construction also properties of the tangent and normal are deducible.

Complete the rectangle $QAEB$; then, since the directions of motion of A and B are respectively perpendicular to EA and EB, the state of motion of the line AB may be represented by supposing that the triangle EAB is turning round the point E.

Hence it follows that EP is the normal to the locus of P and that PT perpendicular to EP is the tangent.

Let OF parallel to PT meet EP in F; then since a circle can be drawn through $OFBE$,

the angle $PFB = EOB = PBG$,

and the triangles PGB, PFB are similar.

Hence $PG : PB :: PB : PF$,

or $PG \cdot PF = PB^2$,

where PB is equal to the semi-conjugate axis; and, similarly, by joining AF, it can be shewn that $Pg \, . \, PF = AP^2$, g being the point of intersection of PG and AO.

Again, a circle can be drawn round $EQPB$, and, since EPT, EBT are right angles, T is on this circle, and therefore TQE is a right angle.

Hence OQN, and OQT are similar triangles, and

$$ON : OQ :: OQ : OT,$$

or

$$ON \, . \, OT = AP^2,$$

where AP is equal to the semi-transverse axis.

249. Observing that the circle circumscribing the rectangle passes through F, we have

$$PF \, . \, PE = AP \, . \, PB;$$

hence PE is equal to the semi-diameter conjugate to OP.

This suggests another construction for the problem solved in Art. (216).

Thus, if OP, OD be the given semi-conjugate diameters, draw PF perpendicular to OD, and in FP produced, take PE equal to OD.

Join OE, bisect it in C, and take CQ equal to CP;

then OA, OB drawn parallel and perpendicular to PQ, and meeting CP in A and B, will be the directions of the axes, and their lengths will be equal to AP and PB.

MISCELLANEOUS PROBLEMS.

1. Given two fixed points of a conic and the directrix, find the locus of the focus.

2. A line cuts two concentric, similar and similarly situated ellipses in P, Q, q, p. If the line move parallel to itself, $PQ \cdot Qp$ is constant.

3. If the ordinate NP of a conic be produced so that $NQ=SP$, find the locus of Q.

4. If a circle be described passing through any point P of a given hyperbola, and the extremities of the transverse axis, and the ordinate NP be produced to meet the circle in Q, the locus of Q is an hyperbola.

5. PQ is one of a series of chords inclined at a constant angle to the diameter AB of a circle; find the locus of the intersection of AP, BQ.

6. If from a point T in the director circle of an ellipse, tangents TP, TP' be drawn, the line joining T with the intersection of the normals at P and P' passes through the centre.

7. The ordinate NP at any point of an ellipse is produced to a point Q, such that NQ is equal to the subtangent at P; prove that the locus of Q is an hyperbola.

8. OP, OQ touch a parabola at the points P, Q; another line touches the parabola in R, and meets OP, OQ in S, T; if V be the intersection of the lines PT, SQ, O, R, V are in a straight line.

9. On all parallel chords of a circle a series of isosceles triangles are described, having the same vertical angle, and having their planes perpendicular to the plane of the circle. Find the locus of their vertices; and find what the vertical angle must be in order that the locus may be a circle.

10. AK, BL are two parallel straight lines, AB perpendicular to both of them; take any point Q in BL, join AQ; in AQ, produced if necessary, take a point P, such that if PN be drawn perpendicular to AK, PN is equal to BQ: shew that the locus of P is a parabola.

11. A circle always passes through a fixed point, and cuts a given straight line at a constant angle, prove that the locus of its centre is an hyperbola.

12. The diameter, through a point P, of a parabola, meets the tangent at the vertex in Z; the normal at P and the focal distance of Z will intersect in a point at the same distance from the tangent at the vertex as P.

13. S and H being the foci, P a point in the ellipse, if SH be bisected in L, and AL be drawn from the vertex cutting SP in Q, the locus of Q is an ellipse whose focus is S.

14. If normals be drawn at the ends of focal chords of a parabola, the locus of their point of intersection is a parabola.

15. If the diagonals of a quadrilateral circumscribing an ellipse meet in the centre the quadrilateral is a parallelogram.

16. A series of ellipses pass through the same point, and have a common focus, and their major axes of the same length; prove that the locus of their centres is a circle.

What are the limits of the eccentricities of the ellipses, and what does the ellipse become at the higher limit?

17. If PSp be a focal chord of a conic, Q any point of the conic, and if PQ, pQ meet the directrix in D and E, DSE is a right angle.

18. Tangents are drawn to an ellipse from a point on a similar and similarly situated concentric ellipse; shew that if P, Q be the points of contact, A, A' the ends of the axis of the first ellipse, the loci of the intersections of AP, $A'Q$, and of AQ, $A'P$ are two ellipses similar to the given ellipses.

19. If TP, TQ be two tangents to a parabola, such that the chord PQ is normal at P,

$$PQ : PT :: PN : AN,$$

PN and AN being ordinate and abscissa of P.

20. Draw a parabola which shall touch four given straight lines.

Under what condition is it possible to describe a parabola touching five given straight lines?

21. Two equal parabolas have a common focus; and, from any point in the common tangent, another tangent is drawn to each: prove that these tangents are equidistant from the common focus.

22. Two parabolas have a common axis and vertex, and their concavities turned in opposite directions, the latus rectum of one is eight times that of the other; prove that the portion of a tangent to the former, intercepted between the common tangent and axis, is bisected by the latter.

23. The locus of the intersections of the focal distances of any point on an ellipse with the central perpendicular upon the tangent at the same point consists of two circles.

24. Given two points, a focus, and the eccentricity; describe the conic.

25. PSp is a focal chord of a parabola, RD is the directrix meeting the axis in D, Q is any point in the curve; prove that if QP, Qp produced meet the directrix in R, r, half the latus rectum will be a mean proportional between DR, Dr.

26. PQ is any chord of a parabola cutting the axis in L, R, R' are the two points in the parabola at which this chord subtends a right angle: if RR' be joined, meeting the axis in L', LL' will be equal to the latus-rectum.

27. If the internal and external bisectors of the angle between two tangents to a central conic meet the transverse axis in G and T, shew that $CG \,.\, CT$ is constant.

28. If the tangent at any point P of an hyperbola cut an asymptote in T, and if SP cut the same asymptote in Q, then $SQ = QT$.

29. Tangents to an ellipse are drawn from any point in a circle through the foci, prove that the lines bisecting the angle between the tangents, all pass through a fixed point.

30. PN is a semi-ordinate of a parabola, and AM is taken on the other side of the vertex along the axis equal to AN; from

any point Q in PN, QR is drawn parallel to the axis meeting the curve in R; shew that the lines MR, AQ will intersect in the parabola.

31. The volume of a cone cut off by a plane bears a constant ratio to the cube, the edge of which is equal to the minor axis of the section.

32. A tangent to an ellipse at P meets the minor axis in t, and tQ is perpendicular to SP; prove that SQ is of constant length, and that if PM be the perpendicular on the minor axis, QM will meet the major axis in a fixed point.

33. Describe an ellipse with a given focus touching three given straight lines, no two of which are parallel and on the same side of the focus.

34. Prove that the conic which touches the sides of a triangle, and has its centre at the centre of the nine-point circle, has one focus at the orthocentre, and the other at the centre of the circumscribing circle.

35. Given a chord of a parabola in magnitude and position and the point in which the axis cuts the chord, the locus of the vertex is a circle.

36. From Q the middle point of a chord PP' of an ellipse whose focus is S, QG is drawn perpendicular to PP' to meet the major axis in G; prove that $2 . SG : SP+SP' :: SA : AX$.

37. An ellipse and parabola have the same focus and directrix; tangents are drawn to the ellipse at the extremities of the major axis: shew that the diagonals of the quadrilateral formed by the four points where these tangents cut the parabola intersect in the common focus, and pass through the extremities of the minor axis of the ellipse.

38. A straight rod moves in any manner in a plane; prove that, at any instant, the directions of motion of all its particles are tangents to a parabola.

39. If from a point T on the auxiliary circle, two tangents be drawn to an ellipse touching it in P and Q, and when produced meeting the circle again in p, q; shew that the angles PSp and QSq are together equal to the supplement of PTQ.

40. Tangents at the extremities of a pair of conjugate diameters of an ellipse meet in T; prove that ST, $S'T$ meet the conjugate diameters in four points which lie on a circle.

41. From a point T in the auxiliary circle tangents are drawn to an ellipse, touching it in P and Q, and meeting the auxiliary circle again in p and q; shew that the angle pCq is equal to the sum of the angles PSQ and $PS'Q$.

42. The angle between the focal distance and tangent at any point of an ellipse is half the angle subtended at the focus by the diameter through the point.

43. Tangents to an ellipse, foci S and H, at the ends of a focal chord PHP' meet the further directrix in Q, Q'. The parabola, whose focus is S, and directrix PP', touches PQ, $P'Q'$, in Q, Q'; it also touches the normals at P, P', and the minor axis, and has for the tangent at its vertex the diameter parallel to PP'.

44. S is a fixed point, and E a point moving on the arc of a given circle; prove that the envelope of the straight line through E at right angles to SE is a conic.

45. A circle passing through a fixed point S cuts a fixed circle in P, and has its centre at O; the lines which bisect the angle SOP all touch a conic of which S is a focus.

46. Through a given point within a parabola draw a chord which shall be divided in a given ratio in that point.

47. If a chord $RPQV$ meet the directrices of an ellipse in R and V, and the circumference in P and Q, then RP and QV subtend, each at the focus nearer to it, angles of which the sum is equal to the angle between the tangents at P and Q.

48. The tangent to an ellipse at P meets the directrix, corresponding to S, in Z: through Z a straight line ZQR is drawn cutting the ellipse in Q, R; and the tangents at Q, R intersect (on SP) in T. Shew that a conic can be described with focus S, and directrix PZ, to pass through Q, R and T; and that TZ will be the tangent at T.

49. A sphere is described about the vertex of a right cone as centre; the latera recta of all sections made by tangent planes to the sphere are equal.

50. TP, TQ are tangents to an ellipse at P and Q; one circle touches TP at P and meets TQ in Q and Q', another touches TQ at Q and meets TP in P and P'; prove that PQ' and QP' are divided in the same ratio by the ellipse.

51. If the tangent at any point P cut the axes of the curve, produced if necessary, in T and T', and if C be the centre of the curve, prove that the area of the triangle TCT' varies inversely as the area of the triangle PCN, where PN is the ordinate of P.

52. The circle of curvature of an ellipse at P passes through the focus S, SM is drawn parallel to the tangent at P to meet the diameter of the ellipse in M; shew that it divides this diameter in the ratio of $3 : 1$.

53. Prove the following construction for a pair of tangents from any external point T to an ellipse of which the centre is C: join CT, let $TPCP'T$ a similar and similarly situated ellipse be drawn, of which CT is a diameter, and P, P' its points of intersection with the given ellipse; TP, TP' will be tangents to the given ellipse.

54. In an ellipse SR, $S'R$, drawn perpendicularly to a pair of conjugate diameters, intersect in R; prove that the locus of R is a concentric ellipse.

55. An ellipse and confocal hyperbola intersect in P: prove that an asymptote of the hyperbola passes through the point on the auxiliary circle of the ellipse corresponding to P.

56. Through a fixed point a pair of chords of a circle are drawn at right angles: prove that each side of the quadrilateral formed by joining their extremities envelopes a conic of which the fixed point and the centre of the circle are foci.

57. A conic section is circumscribed by a quadrilateral $ABCD$: A is joined to the points of contact of CB, CD; and C to the points of contact of AB, AD: prove that BD is a diagonal of the interior quadrilateral thus formed.

58. Any conic passing through the four points of intersection of two rectangular hyperbolas will be itself a rectangular hyperbola.

59. A focal chord PSQ is drawn to a conic of which C is the centre; the tangents and normals at P and Q intersect in T and K respectively; shew that ST, SP, SK, SC form an harmonic pencil.

60. If two focal chords be drawn in a rectangular hyperbola parallel to conjugate diameters the lines joining their extremities will intersect on the asymptotes.

61. R is the middle point of a chord PQ of a rectangular hyperbola whose centre is C. Through R, RQ', RP' are drawn parallel to the tangents at P and Q respectively, meeting CQ, CP in Q', P'. Prove that a circle can be described about C, P', R, Q'.

62. Common tangents are drawn to two parabolas, which have a common directrix, and intersect in P, Q: prove that the chords joining the points of contact in each parabola are parallel to PQ, and the part of each tangent between its points of contact with the two curves is bisected by PQ produced.

63. An ellipse has its centre on a given hyperbola and touches the asymptotes. The area of the ellipse being always a maximum prove that its chord of contact with the asymptotes always touches a similar hyperbola.

64. A circle and parabola have the same vertex A and a common axis. $BA'C$ is the double ordinate of the parabola which touches the circle at A', the other extremity of the diameter which passes through A; PP' any other ordinate of the parabola parallel to this, meeting the axis in N and the chord AB produced in R: shew that the rectangle between RP and RP' is proportional to the square on the tangent drawn from N to the circle.

65. If two confocal conics intersect, prove that the centre of curvature of either curve at a point of intersection is the pole of the tangent at that point with regard to the other curve.

66. Tangents are drawn at two points P, P' on an ellipse. If any tangent be drawn meeting those at P, P' in R, R' shew that the line bisecting the angle RSR' intersects RR' on a fixed tangent to the ellipse. Find the point of contact of this tangent.

67. Having given a focus and two tangents to a conic, shew that the chord of contact passes through a fixed point.

68. Having given a pair of conjugate diameters of an ellipse, PCP', DCD', let PF be the perpendicular from P on CD, in PF take PE equal to CD, bisect CE in O, and on CE as diameter describe a circle; prove that PO will meet the circle in two points Q and R such that CQ, CR are the directions of the semi-axes, and PQ, PR their lengths.

69. If from any point A a straight line AEK be drawn parallel to an asymptote of an hyperbola, and meeting the polar of A in K and the curve in E, shew that $AE=EK$.

70. Two cones, whose vertical angles are supplementary, are placed with their vertices coincident and their axes at right angles, and are cut by a plane perpendicular to a common generating line; prove that the directrices of the section of one cone pass through the foci of the section of the other.

71. PQ is a normal at a point P of a rectangular hyperbola meeting the curve again in Q, shew that PQ is equal to the diameter of curvature at P.

72. The normal at a point P of an ellipse meets the curve again in P', and through O, the centre of curvature at P, the chord QOQ' is drawn at right angles to PP'; prove that

$$QO \,.\, OQ' : PO \,.\, OP' :: 2 \,.\, PO : PP'.$$

73. From an external point T, tangents are drawn to an ellipse, the point of contact being on the same side of the major axis. If the focal distances of these points intersect in M and N, TM, TN are tangents to a confocal hyperbola, which passes through M and N.

74. A point moves in a plane so that the sum or difference of its distances from two fixed points, one in the given plane and the other external to it, is constant. It will describe a conic, the section of a right cone whose vertex is the given external point.

75. If a triangle is self-conjugate with respect to each of a series of parabolas, the lines joining the middle points of its sides will be tangents: all the directrices will pass through O, the centre of the circumscribing circle: and the focal chords, which are the polars of O, will envelope an ellipse inscribed in the given triangle which has the nine-point circle for its auxiliary circle.

76. If PP' be a double ordinate of an ellipse, and if the normal at P meet CP' in O, prove that the locus of O is a similar ellipse, and that the square on its axis is to that on the axis of the given ellipse in the ratio

$$AC^2 - BC^2 : AC^2 + BC^2.$$

77. If SY, HZ be focal perpendiculars on the tangent at P of an ellipse, and SY', HZ' perpendiculars on the tangents from P to a confocal ellipse, prove that the rectangle from $YY' \,.\, ZZ'$ is equal to the difference of the squares on the semi-axes.

78. Two points describe the circumference of an ellipse, with velocities which are to one another in the ratio of the squares on the diameters parallel to their respective directions of motion. Prove that the locus of the point of intersection of their directions of motion will be an ellipse, confocal with the given one.

79. By help of the geometry of the cone, or otherwise, prove that the sum of the tangents from any point of an ellipse to the circles of curvature at the vertices is constant.

80. Three chords of a circle are drawn through a point on the circumference; with this point as focus and the chords as axes three parabolas are described whose parameters are inversely proportional to the chords. Prove that the common tangents to the parabolas, taken two and two, meet in a point.

81. A parabola touches the three lines BC, CA, AB in P, Q, R, and through R a line parallel to the axis meets RQ in E; shew that $ABEC$ is a parallelogram.

82. If a conic be inscribed in a quadrilateral, shew that the locus of its centre is a straight line.

Shew also that this line passes through the middle points of the diagonals.

83. Any conic passing through four points has a pair of conjugate diameters parallel to the axes of the two parabolas which can be drawn through the four points.

84. If two circles be inscribed in a conic, and tangents be drawn to the circles from any point in the conic, the sum or difference of these tangents is constant, according as the point does or does not lie between the two chords of contact.

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